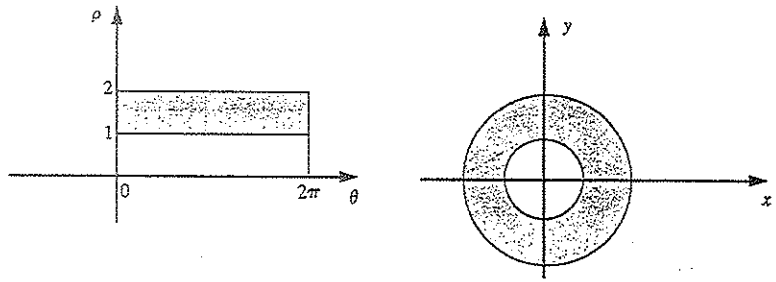
CAPÍTULO 1

Exercícios 1.1

1. Sejam $\varphi(\theta, \rho) = (x, y)$, com $x = \rho \cos \theta$ e $y = \rho \operatorname{sen} \theta$.

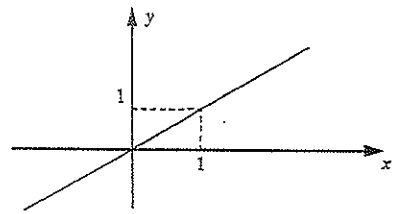


Para cada ρ fixo, $1 \leq \rho \leq 2$, φ transforma o segmento (θ, ρ), $0 \leq \theta \leq 2\pi$, na circunferência $x^2 + y^2 = (\rho \cos \theta)^2 + (\rho \operatorname{sen} \theta)^2 = \rho^2$. Assim, φ transforma o retângulo $0 \leq \theta \leq 2\pi$, $1 \leq \rho \leq 2$ na coroa circular $1 \leq x^2 + y^2 \leq 4$.

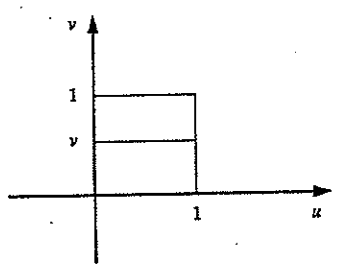
2. Sejam $\varphi(u, v) = (x, y)$, $x = u + v$, $y = u - v$.

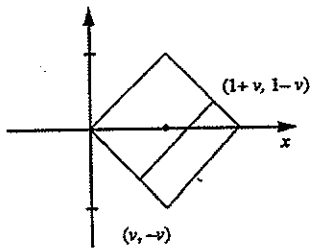
a) $B = \{(u, 0) \in \mathbb{R}^2\}$
$\varphi(B) = \{(x, y) \in \mathbb{R}^2 \mid x = y\}$

b) $B = \{(u, v) \in \mathbb{R}^2 \mid 0 \leq u \leq 1, 0 \leq v \leq 1\}$.



Para cada v fixo, $0 \leq v \leq 1$, φ transforma o segmento (u, v), $0 \leq u \leq 1$, no segmento $(x, y) = (u + v, u - v)$, $0 \leq u \leq 1$, de extremidades $(v, -v)$ e $(1 + v, 1 - v)$.





$\varphi(B)$ é o quadrado de vértices (0, 0), (1, −1), (2, 0), (1, 1).

3. Seja $B = \{(u, v) \mid u^2 + v^2 \leq r^2\}$.

$\varphi(u, v) = (x, y) = (u + v, u - v)$.

Temos:

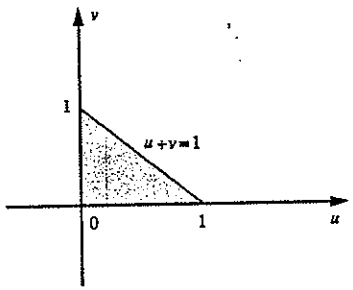
$$x^2 + y^2 = (u+v)^2 + (u-v)^2 = 2u^2 + 2v^2 = \underbrace{2(u^2+v^2)}_{\leq r^2}$$

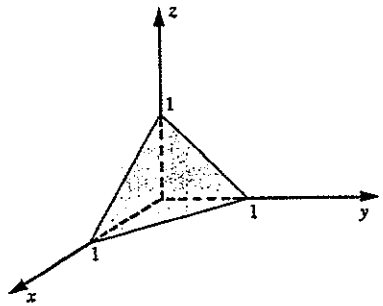
Portanto,

$\varphi(B) = \{(x, y) \mid x^2 + y^2 \leq 2r^2\}$.

5. Seja $f(u, v) = (u, v, 1 - u - v)$ com $u \leq 0$, $v \geq 0$ e $u + v \leq 1$

A imagem de f coincide com o gráfico da função $z = 1 - x - y$, $x \geq 0$, $y \geq 0$ e $x + y \leq 1$.





6. Seja $\sigma(u, v) = (x, y, z)$ com $x = u \cos v$, $y = u \operatorname{sen} v$ e $z = u$

a) É a circunferência $x = u_1 \cos v$, $y = u_1 \operatorname{sen} v$ e $z = u_1$ contida no plano $z = u_1$ e com centro no ponto $(0, 0, u_1)$, com u_1 fixo e $v \in \mathbb{R}$.

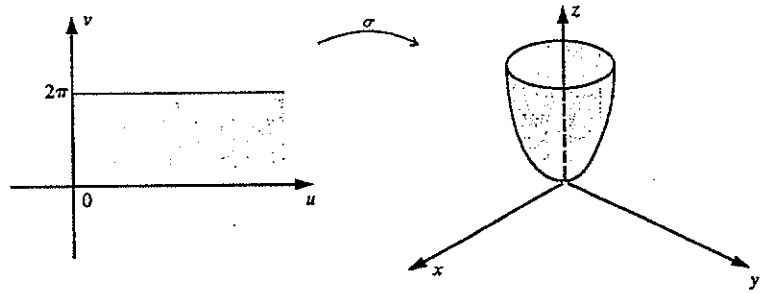
b) É a reta $x = u \cos v_1$, $y = u \operatorname{sen} v_1$ e $z = u$, com v_1 fixo e $u \in \mathbb{R}$.

c) É a superfície lateral do cone circular reto gerada pela rotação, em torno do eixo z, do segmento $x = u \cos v$, $y = u \operatorname{sen} v$ e $z = u$, $0 \leq u \leq 1$ e $0 \leq v \leq 2\pi$, v fixo.

7. Seja $\sigma(u, v) = (x, y, z)$ com $x = u \cos v$, $y = u \operatorname{sen} v$ e $z = u^2$.

Temos $x^2 + y^2 = u^2 \cos^2 v + u^2 \operatorname{sen}^2 v = u^2 \Rightarrow x^2 + y^2 = z$

$\operatorname{Im}(\sigma) = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = z\}$ que é um parabolóide



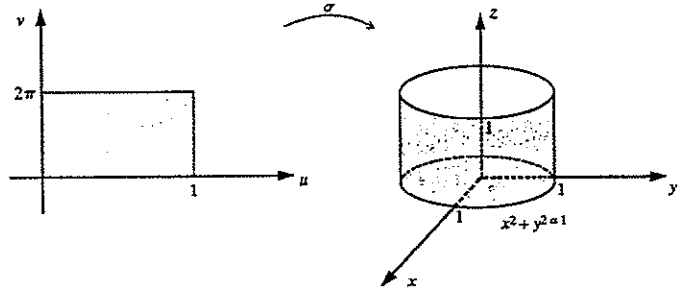
8. Seja $\sigma(u, v) = (\cos v, \operatorname{sen} v, u)$, com $0 \leq u \leq 1$ e $0 \leq v \leq 2\pi$. Temos:

$$\left.\begin{array}{l} x = \cos v \\ y = \operatorname{sen} v \end{array}\right\} \Rightarrow x^2 + y^2 = 1$$

e

$z = u \Rightarrow 0 \leq z \leq 1$.

$\operatorname{Im}(\sigma) = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1 \text{ e } 0 \leq z \leq 1\}$ é uma superfície cilíndrica.

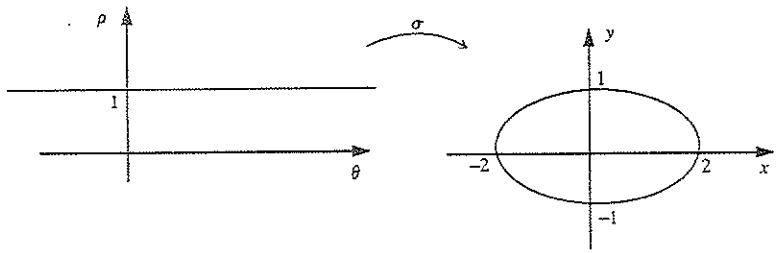


10. Seja $\sigma(\theta, \rho) = (2\rho \cos \theta, \rho \operatorname{sen} \theta)$.

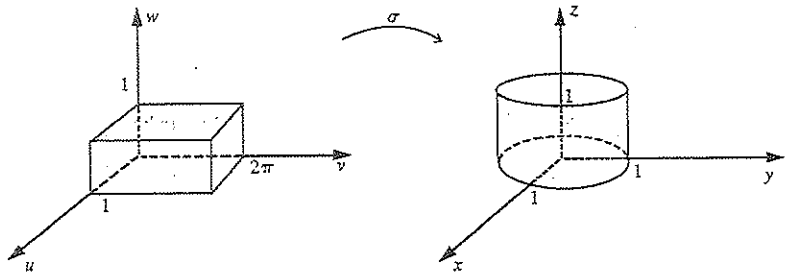
$$\begin{cases} x = 2\rho \cos\theta \\ y = \rho \operatorname{sen}\theta \end{cases} \Rightarrow \underbrace{\cos^2\theta + \operatorname{sen}^2\theta}_{1} = \frac{x^2}{4\rho^2} + \frac{y^2}{\rho^2}$$

Quando o ponto $(\theta, 1)$ descreve a reta $\rho = 1$, o ponto (x, y) descreve a elipse

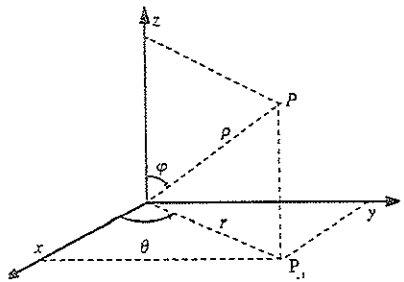
$$\frac{x^2}{4} + \frac{y^2}{1} = 1$$



12. Seja $\sigma(u, v, w) = (u \cos v, u \operatorname{sen} v, w)$, com $0 \leq u \leq 1$, $0 \leq v \leq 2\pi$ e $0 \leq w \leq 1$.
Temos $x = u \cos v$ e $y = u \operatorname{sen} v$. Então, $x^2 + y^2 = u^2$ $(0 \leq u \leq 1)$
$\operatorname{Im}(\sigma) = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 \leq 1 \text{ e } 0 \leq z \leq 1\}$ é um cilindro.



14.



Temos:

$r = \rho \cos(\frac{\pi}{2} - \phi) = \rho \operatorname{sen} \varphi,$
$x = r \cos \theta = \rho \operatorname{sen} \varphi \cos \theta,$
$y = r \operatorname{sen} \theta = \rho \operatorname{sen} \varphi \operatorname{sen} \theta,$
$z = \rho \cos \varphi,$

com $\rho \geq 0$, $0 \leq \theta \leq 2\pi$ e $0 \leq \phi \leq \pi$.

15. *a*) Em coordenadas esféricas:

$x = \rho_1 \operatorname{sen} \varphi \cos \theta$, $y = \rho_1 \operatorname{sen} \varphi \operatorname{sen} \theta$ e $z = \rho_1 \cos \varphi$. Temos:

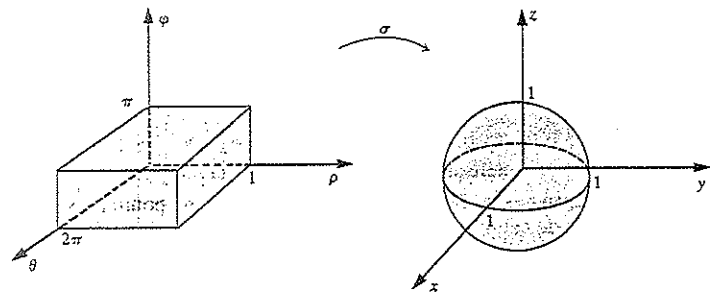
$$\begin{aligned} x^2 + y^2 + z^2 &= \rho_1^2 \operatorname{sen}^2 \varphi \cos^2 \theta + \rho_1^2 \operatorname{sen}^2 \varphi \operatorname{sen}^2 \theta + \rho_1^2 \cos^2 \varphi \\ &= \rho_1^2 \operatorname{sen}^2 \varphi (\cos^2 \theta + \operatorname{sen}^2 \theta) + \rho_1^2 \cos^2 \varphi \\ &= \rho_1^2 (\operatorname{sen}^2 \varphi + \cos^2 \varphi). \quad \Rightarrow \quad x^2 + y^2 + z^2 = \rho_1^2 \end{aligned}$$

E, portanto, $x^2 + y^2 + z^2 = \rho_1^2$.

$\varphi(B) = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = \rho_1^2\}$ é uma superfície esférica de raio ρ_1.

b) Seja $B = \{(\theta, \rho, \varphi) \in \mathbb{R}^3 \mid 0 \leq \rho \leq 1, 0 \leq \theta \leq 2\pi \text{ e } 0 \leq \varphi \leq \pi\}$ um paralelepípedo.

$\sigma(B) = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \leq 1\}$ é a esfera de raio 1.



Exercícios 1.2

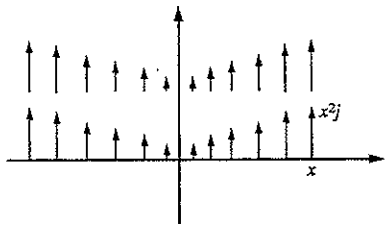
1. *a*) $\vec{v}(x, y) = x^2 \vec{j}$

A função vetorial $\vec{v}$ associa a cada ponto (x, y) do plano o vetor $x^2 \vec{j}$.

A todos os pontos do eixo dos y, $\vec{v}$ associa o vetor nulo, pois $\vec{v}(0, y) = 0^2 \vec{j}$.

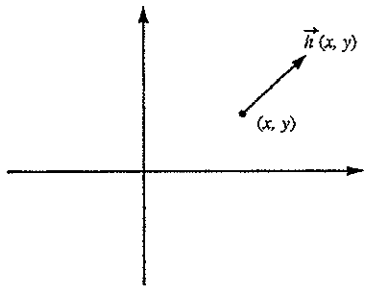
A todos os pontos que estão sobre a reta $x = 1$, $\vec{v}$ associa o vetor $\vec{j}$.

De forma geral, $\vec{v}$ associa a todos os pontos que estão sobre uma mesma reta $x = a$, o vetor $a^2\vec{j}$. A figura mostra este campo:



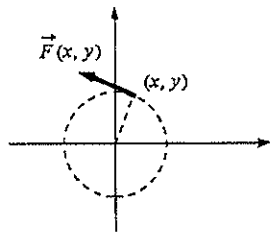
b) $\vec{h}(x, y) = \vec{i} + \vec{j}$

A função vetorial $\vec{h}$ associa a cada ponto (x, y) do plano o vetor $\vec{i} + \vec{j}$. $\|h(x, y)\| = \sqrt{2}$ é a intensidade do campo no ponto (x, y).

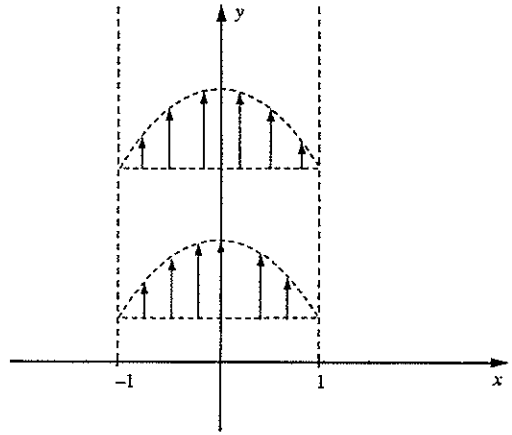


c) $\vec{F}(x, y) = -y\vec{i} + x\vec{j}$.

$\|\vec{F}(x, y)\| = \sqrt{y^2 + x^2}$. A intensidade do campo é a mesma nos pontos de uma mesma circunferência de centro na origem. A intensidade do campo no ponto (x, y) é igual ao raio da circunferência de centro na origem e que passa por este ponto. Além disso, $\vec{F}(x, y)$ é perpendicular ao vetor $x\vec{i} + y\vec{j}$ de posição do ponto (x, y), pois, $(x\vec{i} + y\vec{j}) \cdot (-y\vec{i} + x\vec{j}) = 0$, logo, $\vec{F}(x, y)$ é tangente em (x, y) a tal circunferência.

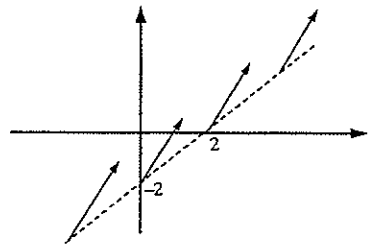


d) $\vec{v}(x, y) = \underbrace{(1 - x^2)}_{>0}\vec{j}$ com $|x| < 1$.



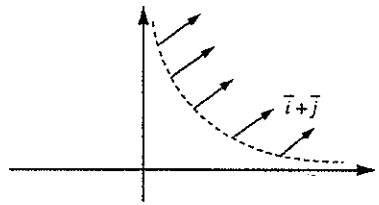
2. $\vec{f} = (x, y) = \vec{i} + (x - y)\vec{j}$.

c) $y = x - 2$, daí $\vec{f}(x, x - 2) = \vec{i} + 2\vec{j}$.



3. $\vec{g}(x, y) = \vec{i} + xy\,\vec{j}$.

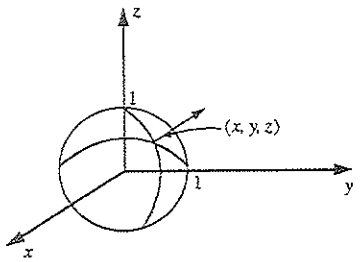
Se $xy = 1$ temos:
$\vec{g}(x, y) = \vec{i} + \vec{j}$



6. $\vec{F} = \nabla f$ onde $f(x, y, z) = x^2 + y^2 + z^2$.

$\nabla f(x, y, z) = (2x, 2y, 2z)$.

Seja (x, y, z) um ponto da superfície esférica $x^2 + y^2 + z^2 = 1$. Neste ponto $\nabla\varphi(x, y, z) = (2x, 2y, 2z)$ é normal à superfície esférica $x^2 + y^2 + z^2 = 1$.



7. Seja $\vec{F} = \nabla f$ onde $f(x, y, z) = x + y + z$. $\underbrace{\vec{F}(x, y, z)}_{\nabla f} = (1, 1, 1)$.

$\vec{F} = \nabla f$ é normal no ponto (x, y, z) à superfície de nível de f que passa por esse ponto.

Como $x + y + z = 1$ é uma superfície de nível de f, então $\vec{F}(x, y, z) = (1, 1, 1)$ é normal, no ponto (x, y, z), ao plano $x + y + z = 1$.

9. Sabemos que $V(x, y) = x^2 + y^2$, $\nabla V(x, y) \cdot \vec{F}(x, y) \geqslant 0$ e $\gamma'(t) = \vec{F}(\gamma(t))$, onde $\gamma(t) = (x(t), y(t)), t \in I$. Primeiro, vamos mostrar que $g(t) = V(\gamma(t))$ é decrescente no intervalo I. De fato, $g'(t) = \nabla V(\gamma(t)) \cdot \gamma'(t) = \nabla V(\gamma(t)) \cdot \vec{F}(\gamma(t)) \leqslant 0$ em I; logo, $g(t)$ é decrescente em I. De $g(t) = V(\gamma(t)) = x^2(t) + y^2(t)$, segue que $g(t)$ é o quadrado da distância da origem ao ponto $\gamma(t)$. De $g'(t) \leqslant 0$, segue que tal distância decresce quando t cresce. Assim, se $\gamma(t_0)$ é ponto do círculo $x^2 + y^2 \leqslant r^2$, então, para $t \geqslant t_0$, $\gamma(t)$ permanecerá em tal círculo.

10. *a*) De $g'(t) = \nabla V(\gamma(t)) \cdot \vec{F}(\gamma(t)) < 0$, segue que g é estritamente decrescente em $[0, +\infty[$, ou seja, a distância do ponto $\gamma(t)$ à origem é estritamente decrescente. Observe que, para $t \to +\infty$, duas situações poderão ocorrer: ou $\gamma(t)$ tende para a origem ou $\gamma(t)$ irá se aproximando mais e mais de uma circunferência. De acordo?

b) Como $\nabla V(x, y) \cdot \vec{F}(x, y)$ é contínua e estritamente menor que zero no compacto $r^2 \leqslant x^2 + y^2 \leqslant R^2$, segue que $M < 0$. Temos

$$g'(t) = \nabla V(\gamma(t)) \cdot \gamma'(t) \leqslant M \text{ em } [0, T].$$

Daí

$$g(t) - g(0) = \int_0^t \nabla V\gamma(t) \cdot \gamma'(t)dt \leqslant dt \leqslant Mt,\ 0 \leqslant t \leqslant T$$

e, portanto, para t em $[0, T]$, $V(\gamma(t)) - V(\gamma(0)) \leqslant Mt$.

c) Se $\gamma(t)$ permanecesse na coroa $r^2 \leqslant x^2$ 1 $y^2 \leqslant R^2$ para todo $t > 0$, a desigualdade acima seria verdadeira para todo $t \geqslant 0$, que é absurda, pois $\lim_{t \to +\infty} Mt = -\infty$ e $V(\gamma(t)) \geqslant 0$.

d) De $g'(t) \leqslant 0$ pata $t \geqslant 0$, segue que $g(t)$ é estritamente decrescente em $[0, +\infty[$, além disto, $g(t) = V(\gamma(t)) \geqslant 0$ para todo $t \geqslant 0$. Logo, $\lim_{t \to +\infty} V(\gamma(t))$ existe e é um número $L \geqslant 0$. Se tivéssemos $L > 0$, $\gamma(t)$ permaneceria em uma coroa $r^2 \leqslant x^2 + y^2 \leqslant R^2$, com $0 < r < R$, que contradiz o que vimos em *c*). Logo, $L = 0$.

e) Dado $\varepsilon > 0$, existe $\delta > 0$, tal que, para todo $t > \delta$, $\|\gamma(t)\| < \varepsilon$ (ou seja, para $t > \delta$, $\gamma(t)$ permanece fora da coroa $\varepsilon^2 < x^2 + y^2 < R^2$), logo, $\lim_{t \to +\infty} \gamma(t) = (0, 0)$.

11. Tomando-se $V(x, y) = x^2 + y^2$ e $\vec{F}(x, y) = (-y - x^3)\vec{i} + (x - y^3)\vec{j}$ obtemos $\nabla V(x, y) \cdot \vec{F}(x, y) = (2x, 2y) \cdot (-y - x^3, x - y^3)$, ou seja,
$\nabla V(x, y) \cdot \vec{F}(x, y) = -2(x^4 + y^4) < 0$ para $(x, y) \neq (0, 0)$. Agora, é só aplicar o item *e*) do Exercício 10.

Exercícios 1.3

1. *a*) $\vec{F}(x, y, z) = -y\vec{i} + x\vec{j} + z\vec{k}$.
Temos $P(x, y, z) = -y$, $Q(x, y, z) = x$ e $R(x, y, z) = z$.

$$\text{rot } \vec{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}$$

ou seja rot $\vec{F} = 2\vec{k}$.

d) $\vec{F}(x, y) = (x^2 + y^2)\vec{i}$. De $P(x, y) = x^2 + y^2$ e $Q(x, y) = 0$ segue

rot $\vec{F} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}$, ou seja, rot $\vec{F} = -2y\,\vec{k}$.

2. $g(x, y) = f(\|\vec{r}\|)\vec{r}$ onde f é derivável e $\vec{r} = x\vec{i} + y\vec{j}$.

Temos $g(x, y) = xf(u)\vec{i} + yf(u)\vec{j}$ onde $u = \sqrt{x^2 + y^2}$.

$P(x, y) = xf(u)$ e $Q(x, y) = yf(u)$. Temos

$\frac{\partial P}{\partial y} = xf'(u)\frac{\partial u}{vy} = \frac{xy\,f'(u)}{\sqrt{x^2+y^2}}$ e $\frac{\partial Q}{\partial x} = yf'(u)\frac{\partial u}{\partial x} = \frac{xy\,f'(u)}{\sqrt{x^2+y^2}}$.

Então,

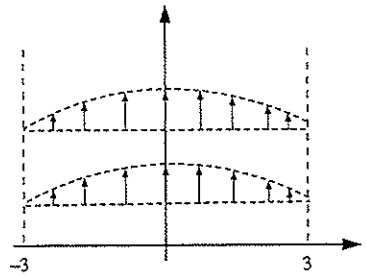
$\text{rot}\,\vec{g} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k} = \vec{0}$.

3. $\vec{F} = \nabla\varphi = \frac{\partial\varphi}{\partial x}\vec{i} + \frac{\partial\varphi}{\partial y}\vec{j}$.

$P(x, y) = \frac{\partial\varphi}{\partial x}$ e $Q(x, y) = \frac{\partial\varphi}{\partial y}$. Então,

$\text{rot}\,\vec{F} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k} = \left(\frac{\partial^2\varphi}{\partial x\partial y} - \frac{\partial\varphi}{\partial y\partial x}\right)\vec{k} = \vec{0}$ (φ é de classe C^2).

$\text{rot}\,\vec{F} = \vec{0} \Leftrightarrow \vec{F}$ é irrotacional.



4. *a*) $\vec{v}(x, y) = \left(1 - \frac{x^2}{9}\right)\vec{j}$.

b) O escoamento é irrotacional pois $\text{rot}\,\vec{v} = \vec{0}$ na região Ω.

6. $\vec{v}(x, y) = -\underbrace{\frac{9y}{(x^2+y^2)^2}}_{P(x,y)}\vec{i} + \underbrace{\frac{x}{(x^2+y^2)^2}}_{Q(x,y)}\vec{j}, \alpha > 0$.

$\text{rot}\,\vec{v}(x, y) = \left(\frac{\partial Q}{\partial x}(x, y) - \frac{\partial P}{\partial y}(x, y)\right)\vec{k}$.

Temos

$\frac{\partial Q}{\partial x}(x, y) = \frac{(1-2\alpha)x^2 + y^2}{(x^2+y^2)^{\alpha+1}}$ e $\frac{\partial P}{\partial y}(x, y) = \frac{-x^2 + (2\alpha-1)y^2}{(x^2+y^2)^{\alpha+1}}$.

Então,

$\frac{\partial Q}{\partial x}(x, y) - \frac{\partial P}{\partial y}(x, y) = \frac{2(1-\alpha)}{(x^2+y^2)^{\alpha}}$.

Segue que para $\alpha \neq 1$, teremos $\text{rot}\,\vec{v}(x, y) \neq 0$.

7. *a*) Seja $\vec{F} = P\vec{i} + Q\vec{j}$, P e Q diferenciáveis.

Consideremos:

$\vec{u} = \cos\alpha\,\vec{i} + \text{sen}\,\alpha\,\vec{j}$ e $\vec{v} = -\text{sen}\,\alpha\,\vec{i} + \cos\alpha\,\vec{j}$, onde $\alpha \in \mathbb{R}$ e $\alpha \neq 0$.

De $(x, y) = s\,\vec{u} + t\,\vec{v}$, segue

$(x, y) = (s\cos\alpha - t\,\text{sen}\,\alpha)\,\vec{i} + (s\,\text{sen}\,\alpha + t\cos\alpha)\,\vec{j}$.

Temos: $\vec{i} = \cos\alpha\,\vec{u} - \text{sen}\,\alpha\,\vec{v}$ e $\vec{j} = \text{sen}\,\alpha\,\vec{u} + \cos\alpha\,\vec{v}$.

$\vec{F}(x, y) = P(x, y)\,\vec{i} + Q(x, y)\,\vec{j}$.

$= P(x, y)[\cos\alpha\,\vec{u} - \text{sen}\,\alpha\,\vec{v}] + Q(x, y)[\text{sen}\,\alpha\,\vec{u} + \cos\alpha\,\vec{v}]$.

Daí,

$\vec{F}(x, y) = [P(x, y)\cos\alpha + Q(x, y)\,\text{sen}\,\alpha]\,\vec{u} + [Q(x, y)\cos\alpha - P(x, y)\,\text{sen}\,\alpha]\,\vec{v}$.

b) Sejam $P_1(s, t) = P(x, y)\cos\alpha + Q(x, y)\,\text{sen}\,\alpha$,

$Q_1(s, t) = Q(x, y)\cos\alpha - P(x, y)\,\text{sen}\,\alpha$,

$x = s\cos\alpha - t\,\text{sen}\,\alpha$,

e $y = s\,\text{sen}\,\alpha + t\cos\alpha$.

Temos $\frac{\partial x}{\partial s} = \cos\alpha$, $\frac{\partial x}{\partial t} = -\text{sen}\,\alpha$; $\frac{\partial y}{\partial s} = \text{sen}\,\alpha$ e $\frac{\partial y}{\partial t} = \cos\alpha$.

Segue que:

$\frac{\partial Q_1}{\partial s}(s, t) - \frac{\partial P_1}{\partial t}(s, t) = \frac{\partial}{\partial s}(Q(x, y)\cos\alpha - P(x, y)\,\text{sen}\,\alpha)$

$-\frac{\partial}{\partial t}\left(P(x, y)\cos\alpha + Q(x, y)\,\text{sen}\,\alpha\right) = -\left(\cos\alpha\frac{\partial Q}{\partial x}\frac{\partial x}{\partial s}\right.$

$\left. + \cos\alpha\frac{\partial Q}{\partial y}\frac{\partial y}{\partial s} - \text{sen}\,\alpha\frac{\partial P}{\partial x}\frac{\partial x}{\partial s} - \text{sen}\,\alpha\frac{\partial P}{\partial y}\frac{\partial y}{\partial s}\right) -$

$$-\left(\cos\alpha\,\frac{\partial P}{\partial x}\overbrace{\frac{\partial x}{\partial t}}^{(-\operatorname{sen}\alpha)} + \cos\alpha\,\frac{\partial P}{\partial y}\overbrace{\frac{\partial y}{\partial t}}^{\cos\alpha} + \operatorname{sen}\alpha\,\frac{\partial Q}{\partial x}\overbrace{\frac{\partial x}{\partial t}}^{(-\operatorname{sen}\alpha)} + \operatorname{sen}\alpha\,\frac{\partial Q}{\partial y}\overbrace{\frac{\partial y}{\partial t}}^{\cos\alpha}\right)$$

$$= \underbrace{(\operatorname{sen}^2\alpha + \cos^2\alpha)}_{1}\frac{\partial Q}{\partial x}(x, y) - \underbrace{(\operatorname{sen}^2\alpha + \cos^2\alpha)}_{1}\frac{\partial P}{\partial y}(x, y).$$

Portanto, $\frac{\partial Q_1}{\partial s}(s, t) - \frac{\partial P_1}{\partial t}(s, t) = \frac{\partial Q}{\partial x}(x, y) - \frac{\partial P}{\partial y}(x, y)$. Isto significa que o rotacional é *invariante* por uma rotação de eixos.

Exercícios 1.4

1. *a*) $\vec{v}(x, y) = -y\vec{i} + x\vec{j}$.

$$\operatorname{div}\vec{v} = \frac{\partial(-y)}{\partial x} + \frac{\partial x}{\partial y} = 0.$$

c) $\vec{F}(x, y, z) = (x^2 - y^2)\vec{i} + \operatorname{sen}(x^2 + y^2)\vec{j} + \operatorname{arctg} z\,\vec{k}$.

$$\operatorname{div}\vec{F} = \frac{\partial(x^2 - y^2)}{\partial x} + \frac{\partial(\operatorname{sen}(x^2 + y^2))}{\partial y} + \frac{\partial(\operatorname{arctg} z)}{\partial z}$$

$$\operatorname{div}\vec{F} = 2x + 2y\cos(x^2 + y^2) + \frac{1}{1 + z^2}.$$

d) $\vec{F}(x, y, z) = (x^2 + y^2 + z^2)\operatorname{arctg}(x^2 + y^2 + z^2)\,\vec{k}$

$$\operatorname{div}\vec{F} = \frac{\partial}{\partial z}[(x^2 + y^2 + z^2)\operatorname{arctg}(x^2 + y^2 + z^2)]$$

$$= 2z\operatorname{arctg}(x^2 + y^2 + z^2) + \frac{2z(x^2 + y^2 + z^2)}{1 + (x^2 + y^2 + z^2)^2}$$

$$= 2z\left[\operatorname{arctg}(x^2 + y^2 + z^2) + \frac{(x^2 + y^2 + z^2)}{1 + (x^2 + y^2 + z^2)^2}\right].$$

2. *a*) $\operatorname{div}\vec{F} \neq 0$ *b*) $\operatorname{div}\vec{F} = 0$

3. Seja $\vec{v}(x, y, z) = y\vec{j}$, $y > 0$.

a) Temos $\operatorname{div}\vec{v} = \frac{\partial}{\partial y}(y) = 1$ (densidade do fluido está diminuindo com o tempo pois $\operatorname{div}\vec{v} > 0$).

Pela equação da continuidade $\operatorname{div}\rho\vec{v} + \frac{\partial\rho}{\partial t} = 0$ (ρ é a densidade do fluido). Daí,

$\frac{\partial\rho}{\partial t} = -\operatorname{div}\rho\vec{v}$ (taxa de variação da densidade do fluido).

Como $\operatorname{div}\vec{v} = 1$ então $\rho \neq$ constante. Logo o fluido não é incompressível (o fluido está se expandindo).

b) $\operatorname{div}\rho\vec{v} = 0 \Rightarrow \operatorname{div}(\rho(y)\cdot y)\vec{j} = 0 \Rightarrow \frac{d}{dy}(\rho(y)\cdot y) = 0$

$\Rightarrow \rho(y)\cdot y = k$ (k constante). Daí, $\rho(y) = \frac{k}{y}$, $y > 0$.

c) Sejam $\rho = \rho(y, t)$ a densidade do fluido e $\vec{v}(x, y, z) = y\vec{j}$ a velocidade do fluido. Temos

$$\operatorname{div}\rho\vec{v} = \operatorname{div}[\rho(y\vec{j})] = \frac{\partial}{\partial y}(\rho y) = y\frac{\partial\rho}{\partial y} + \rho.$$

Substituindo na equação da continuidade $\left(\operatorname{div}\rho\vec{v} + \frac{\partial\rho}{\partial t} = 0\right)$ resulta

$$y\frac{\partial\rho}{\partial y} + \rho + \frac{\partial\rho}{\partial t} = 0 \text{ e, portanto, } \frac{\partial\rho}{\partial t} + y\frac{\partial\rho}{\partial y} = -\rho.$$

4. *a*) Suponhamos que $\vec{v}$ derive de um pontencial, isto é, existe $\varphi : \Omega \to \mathbb{R}$ com $\nabla\varphi = \vec{v}$ em Ω.

Temos $\vec{v}(x, y, z) = \frac{\partial\varphi}{\partial x}\vec{i} + \frac{\partial\varphi}{\partial y}\vec{j} + \frac{\partial\varphi}{\partial z}\vec{k}$

As componentes de $\vec{v}$: $\frac{\partial\varphi}{\partial x}$, $\frac{\partial\varphi}{\partial y}$ e $\frac{\partial\varphi}{\partial z}$ são de classe C^1.

Logo, $\frac{\partial^2\varphi}{\partial x\partial y} = \frac{\partial^2\varphi}{\partial y\partial x}$, $\frac{\partial^2\varphi}{\partial x\partial z} = \frac{\partial^2\varphi}{\partial z\partial x}$ e $\frac{\partial^2\varphi}{\partial y\partial z} = \frac{\partial^2\varphi}{\partial z\partial y}$.

$$\operatorname{rot}\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\varphi}{\partial x} & \frac{\partial\varphi}{\partial y} & \frac{\partial\varphi}{\partial z} \end{vmatrix} = \left(\underbrace{\frac{\partial^2\varphi}{\partial y\partial z} - \frac{\partial^2\varphi}{\partial z\partial y}}_{0}\right)\vec{i}$$

$$+ \left(\underbrace{\frac{\partial^2\varphi}{\partial z\partial x} - \frac{\partial^2\varphi}{\partial x\partial z}}_{0}\right)\vec{j} + \left(\underbrace{\frac{\partial^2\varphi}{\partial x\partial y} - \frac{\partial^2\varphi}{\partial y\partial x}}_{0}\right)\vec{k} = \vec{0}.$$

Se $\operatorname{rot}\vec{v} = \vec{0}$, então $\vec{v}$ é irrotacional.

b) Se $\vec{v}$ é incompressível então div $\vec{v} = 0$.

Mas div $\vec{v}$ = div $(\nabla\varphi) = \nabla\,\nabla\varphi = \nabla^2\varphi$.

Logo, $\nabla^2\varphi =$

$$\varphi \quad = \quad \frac{x}{y},\ y>0.$$

$$\frac{\partial\varphi}{\partial x} = \frac{y}{x^2+y^2} \quad \text{e} \quad \frac{\partial^2\varphi}{\partial x^2} = -\frac{2xy}{(x^2+y^2)^2}.$$

$$\frac{\partial\varphi}{\partial y} = -\frac{x}{x^2+y^2} \quad \text{e} \quad \frac{\partial^2\varphi}{\partial y^2} = \frac{2xy}{(x^2+y^2)^2}.$$

$$\nabla^2\varphi = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} = \frac{-2xy+2xy}{(x^2+y^2)^2} = 0.$$

6. *a*) Seja $\varphi(x, y) = f(\underbrace{x^2+y^2}_{u})$ onde $f(u)$ é derivável até 2.ª ordem.

$$\frac{\partial\varphi}{\partial x} = \frac{df}{du}\cdot\frac{\partial u}{\partial x} = 2xf'(u).$$

$$\frac{\partial^2\varphi}{\partial x^2} = \frac{\partial}{\partial x}\left(2x\,f'(u)\right) = 2\,f'(u) + 2x\,\frac{\partial}{\partial x}\left(f'(u)\right)$$

$$= 2\,f'(u) + 2x\,\frac{d}{du}\left(f'(u)\right)\frac{du}{\partial x}. \text{ Daí}$$

$$\frac{\partial^2\varphi}{\partial x^2} = 2f'(u) + 4x^2\,f''(u).$$

Analogamente, $\dfrac{\partial^2\varphi}{\partial y^2} = 2\,f'(u) + 4y^2 f''(u)$.

Temos que $\nabla^2\varphi = 0$. Logo, $2f'(u) + 4x^2 f''(u) + 2f'(u) + 4y^2 f''(u) = 0$.

Daí, $4\,(\underbrace{x^2+y^2}_{u})\,f''(u) = -\,4f'(u)$ e, portanto, $uf''(u) = -f'(u)$, $u>0$.

b) De $uf''(u) = -f'(u)$, $u>0$, e supondo $f'(u)>0$, temos $\dfrac{f''(u)}{f'(u)} = -\dfrac{1}{u}$ e daí

$(\ln f'(u))' = (-\ln u)'$, $u>0$. Segue que $\ln f'(u) = -\ln u + C$, C constante. Temos,

então, $\ln f'(u) = \ln\dfrac{k}{u}$, onde $\ln k = C$. Daí, $f'(u) = \dfrac{k}{u}$ e, portanto, $f(u) = k\ln u + A$

com k e A constantes, resolve o problema.

7. Devemos esperar div $\nabla\varphi \neq 0$, ou seja, $\nabla^2\varphi \neq 0$.

8. *a*) Seja $\vec{F} = P\vec{i} + Q\vec{j}$ com P e Q diferenciáveis

Temos $\vec{u} = \cos\alpha\vec{i} + \operatorname{sen}\alpha\vec{j}$ e $\vec{v} = -\operatorname{sen}\alpha\vec{i} + \cos\alpha\vec{j}$

Daí vem: $\vec{i} = \cos\alpha\vec{u} - \operatorname{sen}\alpha\vec{v}$ e $\vec{j} = \cos\alpha\vec{v} + \operatorname{sen}\alpha\vec{u}$.

Então,

$$\vec{F}(x, y) = P(x, y)\,(\cos\alpha\vec{u} - \operatorname{sen}\alpha\vec{v}) + Q(x, y)\,(\cos\alpha\vec{v} + \operatorname{sen}\alpha\vec{u})$$

$$\vec{F}(x, y) = \underbrace{[P(x, y)\cos\alpha + Q(x, y)\operatorname{sen}\alpha]}_{P_1(s,t)}\vec{u} + \underbrace{[Q(x, y)\cos\alpha - P(x, y)\operatorname{sen}\alpha]}_{Q_1(s,t)}\vec{v}.$$

b) $\vec{F}_1(s, t) = P_1(s, t)\,\vec{u} + Q_1(s, t)\,\vec{v}$

$(x, y) = s\vec{u} + t\vec{v} \Rightarrow x = s\cos\alpha - t\operatorname{sen}\alpha$ e $y = s\operatorname{sen}\alpha + t\cos\alpha$. Temos

$$\frac{\partial x}{\partial s} = \cos\alpha;\ \frac{\partial x}{\partial t} = -\operatorname{sen}\alpha;\ \frac{\partial y}{\partial s} = \operatorname{sen}\alpha;\ \frac{\partial y}{\partial t} = \cos\alpha$$

Agora:

$$\frac{\partial P_1}{\partial s}(s, t) = \frac{\partial}{\partial s}\left[P(x, y)\cos\alpha + Q(x, y)\operatorname{sen}\alpha\right]$$

$$= \cos\alpha\,\frac{\partial P}{\partial x}\overbrace{\frac{\partial x}{\partial s}}^{\cos\alpha} + \cos\alpha\,\frac{\partial P}{\partial y}\overbrace{\frac{\partial y}{\partial s}}^{\operatorname{sen}\alpha} + \operatorname{sen}\alpha\,\frac{\partial Q}{\partial x}\overbrace{\frac{\partial x}{\partial s}}^{\cos\alpha} + \operatorname{sen}\alpha\,\frac{\partial Q}{\partial y}\overbrace{\frac{\partial y}{\partial s}}^{\operatorname{sen}\alpha}$$

$$= \cos^2\alpha\,\frac{\partial P}{\partial x} + \operatorname{sen}\alpha\cos\alpha\,\frac{\partial P}{\partial y} + \operatorname{sen}\alpha\cos\alpha\,\frac{\partial Q}{\partial x} + \operatorname{sen}^2\alpha\,\frac{\partial Q}{\partial y}\ \textcircled{1}$$

$$\frac{\partial Q_1}{\partial t}(s, t) = \frac{\partial}{\partial t}\left[Q(x, y)\cos\alpha - P(x, y)\operatorname{sen}\alpha\right]$$

$$= \cos\alpha\,\frac{\partial Q}{\partial x}\overbrace{\frac{\partial x}{\partial t}}^{-\operatorname{sen}\alpha} + \cos\alpha\,\frac{\partial Q}{\partial y}\overbrace{\frac{\partial y}{\partial t}}^{\cos\alpha} - \operatorname{sen}\alpha\,\frac{\partial P}{\partial x}\overbrace{\frac{\partial x}{\partial t}}^{-\operatorname{sen}\alpha} - \operatorname{sen}\alpha\,\frac{\partial P}{\partial y}\overbrace{\frac{\partial y}{\partial t}}^{\cos\alpha}$$

$$= -\operatorname{sen}\alpha\cos\alpha\,\frac{\partial Q}{\partial x} + \cos^2\alpha\,\frac{\partial Q}{\partial y} + \operatorname{sen}^2\alpha\,\frac{\partial P}{\partial x} - \operatorname{sen}\alpha\cos\alpha\,\frac{\partial P}{\partial y}\ \textcircled{2}$$

Somando ① e ②:

$$\frac{\partial P_1}{\partial s}(s, t) + \frac{\partial Q_1}{\partial t}(s, t) = \underbrace{(\cos^2\alpha + \operatorname{sen}^2\alpha)}_{1}\frac{\partial P}{\partial x}(x, y) + \underbrace{(\operatorname{sen}^2\alpha + \cos^2\alpha)}_{1}\frac{\partial Q}{\partial y}(x, y)$$

$$\frac{\partial P_1}{\partial s}(s, t) + \frac{\partial Q_1}{\partial t}(s, t) = \frac{\partial P}{\partial x}(x, y) + \frac{\partial Q}{\partial y}(x, y).$$

O que significa que o divergente é *invariante* por uma rotação de eixos.

9. Sejam $\vec{u} = P\vec{i} + Q\vec{j} + R\vec{k}$ e $\vec{v} = P_1\vec{i} + Q_1\vec{j} + R_1\vec{k}$.

a) Vamos supor que as componentes de $\vec{u}$ e $\vec{v}$ admitam derivadas parciais. Temos

$$\operatorname{rot}(\vec{u}+\vec{v}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ (P+P_1) & (Q+Q_1) & (R+R_1) \end{vmatrix}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} + \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P_1 & Q_1 & R_1 \end{vmatrix}$$

$$= \operatorname{rot}\vec{u} + \operatorname{rot}\vec{v}.$$

b) Vamos supor que as componentes de $\vec{u}$ e $\vec{v}$ admitam derivadas parciais. Temos

$$\operatorname{div}(\vec{u}+\vec{v}) = \operatorname{div}\left[(P+P_1)\vec{i} + (Q+Q_1)\vec{j} + (R+R_1)\vec{k}\right]$$

$$= \frac{\partial}{\partial x}(P+P_1) + \frac{\partial}{\partial y}(Q+Q_1) + \frac{\partial}{\partial z}(R+R_1)$$

$$= \left(\frac{\partial}{\partial x}P + \frac{\partial}{\partial y}Q + \frac{\partial}{\partial z}R\right) + \left(\frac{\partial P_1}{\partial x} + \frac{\partial Q_1}{\partial y} + \frac{\partial R_1}{\partial z}\right)$$

$$= \operatorname{div}\vec{u} + \operatorname{div}\vec{v}.$$

d) Vamos supor que φ e as componentes de $\vec{u}$ admitam derivadas parciais. Temos

$$\operatorname{rot}\varphi\vec{u} = \operatorname{rot}(\varphi P\vec{i} + \varphi Q\vec{j} + \varphi R\vec{k}) =$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \varphi P & \varphi Q & \varphi R \end{vmatrix} = \left(\frac{\partial(\varphi R)}{\partial y} - \frac{\partial(\varphi Q)}{\partial z}\right)\vec{i} + \left(\frac{\partial(\varphi P)}{\partial z} - \frac{\partial(\varphi R)}{\partial x}\right)\vec{j}$$

$$+ \left(\frac{\partial(\varphi Q)}{\partial x} - \frac{\partial(\varphi P)}{\partial y}\right)\vec{k} = \left[\varphi\frac{\partial R}{\partial y} + \frac{\partial\varphi}{\partial y}R - \varphi\frac{\partial Q}{\partial z} - \frac{\partial\varphi}{\partial z}Q\right]\vec{i}$$

$$+ \left[\varphi\frac{\partial P}{\partial z} + \frac{\partial\varphi}{\partial z}P - \varphi\frac{\partial R}{\partial x} - \frac{\partial\varphi}{\partial x}R\right]\vec{j} + \left[\varphi\frac{\partial Q}{\partial x} + \frac{\partial\varphi}{\partial x}Q - \varphi\frac{\partial P}{\partial y} - \frac{\partial\varphi}{\partial y}P\right]\vec{k}$$

$$= \varphi\left[\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}\right]$$

$$+ \left[\left(\frac{\partial\varphi}{\partial y}R - \frac{\partial\varphi}{\partial z}Q\right)\vec{i} + \left(\frac{\partial\varphi}{\partial z}P - \frac{\partial\varphi}{\partial x}R\right)\vec{j} + \left(\frac{\partial\varphi}{\partial x}Q - \frac{\partial\varphi}{\partial y}P\right)\right] =$$

$$= \varphi\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} + \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial\varphi}{\partial x} & \dfrac{\partial\varphi}{\partial y} & \dfrac{\partial\varphi}{\partial z} \\ P & Q & R \end{vmatrix}$$

$= \varphi \operatorname{rot}\vec{u} + \nabla\varphi \wedge \vec{u}$. Portanto $\operatorname{rot}\varphi\vec{u} = \varphi \operatorname{rot}\vec{u} + \nabla\varphi \wedge \vec{u}$.

f) Vamos supor as componente de $\vec{u}$ de classe C^2 em Ω. Temos

$$\operatorname{rot}(\operatorname{rot}\vec{u}) = \operatorname{rot}\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

$$= \operatorname{rot}\left[\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}\right].$$

Temos, também,

$$\operatorname{rot}\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{i} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{\partial R}{\partial y} - \dfrac{\partial Q}{\partial z} & 0 & 0 \end{vmatrix}$$

$$= \left(\frac{\partial^2 R}{\partial z\partial y} - \frac{\partial^2 Q}{\partial z^2}\right)\vec{j} - \left(\frac{\partial^2 R}{\partial y^2} - \frac{\partial^2 Q}{\partial y\partial z}\right)\vec{k}.$$

De modo análogo,

$$\operatorname{rot}\left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\vec{j} = -\left(\frac{\partial^2 P}{\partial z^2} - \frac{\partial^2 R}{\partial z\partial x}\right)\vec{i} + \left(\frac{\partial^2 P}{\partial x\partial z} - \frac{\partial^2 R}{\partial x^2}\right)\vec{k}$$

e

$$\operatorname{rot}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k} = \left(\frac{\partial^2 Q}{\partial y\partial x} - \frac{\partial^2 P}{\partial y^2}\right)\vec{i} + \left(\frac{\partial^2 Q}{\partial x^2} - \frac{\partial^2 P}{\partial x\partial y}\right)\vec{j}.$$

Segue que

$$\operatorname{rot}(\operatorname{rot}\vec{u}) = \left(\frac{\partial^2 R}{\partial x\partial z} + \frac{\partial^2 Q}{\partial x\partial y} - \frac{\partial^2 P}{\partial y^2} - \frac{\partial^2 P}{\partial z^2}\right)\vec{i} + \ldots$$

$$= \left(\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 Q}{\partial x\partial y} + \frac{\partial^2 R}{\partial x\partial z} - \frac{\partial^2 P}{\partial x^2} - \frac{\partial^2 P}{\partial y^2} - \frac{\partial^2 P}{\partial z^2}\right)\vec{i} + \ldots$$

De

$$\frac{\partial^2 P}{\partial x^2}+\frac{\partial^2 Q}{\partial x\partial y}+\frac{\partial^2 R}{\partial x\partial z}=\frac{\partial}{\partial x}\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\right)$$

$$=\frac{\partial}{\partial x}(\operatorname{div}\vec{u}),$$

$$\nabla^2 P=\frac{\partial^2 P}{\partial x^2}+\frac{\partial^2 P}{\partial y^2}+\frac{\partial^2 P}{\partial z^2}$$

e com procedimento análogo para as demais componentes de rot (rot $\vec{u}$), resulta

$$\operatorname{rot}(\operatorname{rot}\vec{u})=\left(\frac{\partial}{\partial x}(\operatorname{div}\vec{u})-\nabla^2 P\right)\vec{i}$$

$$+\left(\frac{\partial}{\partial y}(\operatorname{div}\vec{u})-\nabla^2 Q\right)\vec{j}$$

$$+\left(\frac{\partial}{\partial z}(\operatorname{div}\vec{u})-\nabla^2 R\right)\vec{k}$$

ou seja,

$\operatorname{rot}(\operatorname{rot}\vec{u})=\nabla\operatorname{div}\vec{u}-\nabla^2\vec{u}$, onde $\nabla^2\vec{u},=(\nabla^2 P,\nabla^2 Q,\nabla^2 R)$.

10. Sejam $\vec{u}=(u_1,u_2,u_3)$, com componentes de classe C^2, e $\vec{w}=(w_1,w_2,w_3)$ campos vetoriais definidos no aberto Ω de $\mathbb{R}^3$.

Vamos provar que se rot $\vec{u}=\vec{w}$ então div $\vec{w}=0$.

Pela propriedade *e*) do Exercício 9, div rot $\vec{u}=0$, logo, div $\vec{w}=0$.

11. Vamos mostrar que div $(\vec{F}\wedge\vec{G})=\vec{G}\cdot(\nabla\wedge\vec{F})-\vec{F}\cdot(\nabla\wedge\vec{G})$

Temos: $\vec{F}\wedge\vec{G}=\begin{vmatrix}\vec{i}&\vec{j}&\vec{k}\\F_1&F_2&F_3\\G_1&G_2&G_3\end{vmatrix}$, onde $\vec{F}=(F_1,F_2,F_3)$ e $\vec{G}=(G_1,G_2,G_3)$,

e cujas componentes admitem derivadas parciais em Ω.

$\vec{F}\wedge\vec{G}=(F_2G_3-F_3G_2)\vec{i}+(F_3G_1-F_1G_3)\vec{j}+(F_1G_2-F_2G_1)\vec{k}$.

$$\operatorname{div}(\vec{F}\wedge\vec{G})=\nabla\cdot(\vec{F}\wedge\vec{G})=\frac{\partial}{\partial x}(F_2G_3-F_3G_2)+\frac{\partial}{\partial y}(F_3G_1-F_1G_3)$$

$$+\frac{\partial}{\partial z}(F_1G_2-F_2G_1)=F_2\frac{\partial G_3}{\partial x}+\frac{\partial F_2}{\partial x}G_3-F_3\frac{\partial G_2}{\partial x}-\frac{\partial F_3}{\partial x}G_2$$

$$+F_3\frac{\partial G_1}{\partial y}+\frac{\partial F_3}{\partial y}G_1-F_1\frac{\partial G_3}{\partial y}-\frac{\partial F_1}{\partial y}G_3+F_1\frac{\partial G_2}{\partial z}+\frac{\partial F_1}{\partial z}G_2$$

$$-F_2\frac{\partial G_1}{\partial z}-\frac{\partial F_2}{\partial z}G_1=G_1\left(\frac{\partial F_3}{\partial y}-\frac{\partial F_2}{\partial z}\right)+G_2\left(\frac{\partial F_1}{\partial z}-\frac{\partial F_3}{\partial x}\right)+$$

$$+G_3\left(\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right)+F_1\left(\frac{\partial G_2}{\partial z}-\frac{\partial G_3}{\partial y}\right)+F_2\left(\frac{\partial G_3}{\partial x}-\frac{\partial G_1}{\partial z}\right)$$

$$+F_3\left(\frac{\partial G_1}{\partial y}-\frac{\partial G_2}{\partial x}\right)=(G_1,G_2,G_3)\cdot\left(\frac{\partial F_3}{\partial y}-\frac{\partial F_2}{\partial z},\frac{\partial F_1}{\partial z}-\frac{\partial F_3}{\partial x},\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right)$$

$$-(F_1,F_2,F_3)\cdot\left(\frac{\partial G_3}{\partial y}-\frac{\partial G_2}{\partial z},\frac{\partial G_1}{\partial z}-\frac{\partial G_3}{\partial x},\frac{\partial G_2}{\partial x}-\frac{\partial G_1}{\partial y}\right)$$

$$=\vec{G}\cdot(\nabla\wedge\vec{F})-\vec{F}\cdot(\nabla\wedge\vec{G}).$$

Então, div $(\vec{F}\wedge\vec{G})=\vec{G}\cdot(\nabla\wedge\vec{F})-\vec{F}\cdot(\nabla\wedge\vec{G})$.

12) Seja $\vec{F}(x,y)=P(x,y)\,\vec{i}+Q(x,y)\,\vec{j}$ com P e Q de classe C^1.

Temos

$P_1(\theta,\rho)=P(x,y)$ e $Q_1(\theta,\rho)=Q(x,y)$ com $x=\rho\cos\theta$ e $y=\rho\operatorname{sen}\theta$.

a) $\dfrac{\partial P_1}{\partial\theta}=\dfrac{\partial P}{\partial x}\dfrac{\partial x}{\partial\theta}+\dfrac{\partial P}{\partial y}\dfrac{\partial y}{\partial\theta}$ ($\frac{\partial x}{\partial\theta}=-\rho\operatorname{sen}\theta$, $\frac{\partial y}{\partial\theta}=\rho\cos\theta$)

$\dfrac{\partial P_1}{\partial\rho}=\dfrac{\partial P}{\partial x}\dfrac{\partial x}{\partial\rho}+\dfrac{\partial P}{\partial y}\dfrac{\partial y}{\partial\rho}$ ($\frac{\partial x}{\partial\rho}=\cos\theta$, $\frac{\partial y}{\partial\rho}=\operatorname{sen}\theta$)

$$\begin{cases}\dfrac{\partial P_1}{\partial\theta}=-\rho\operatorname{sen}\theta\dfrac{\partial P}{\partial x}+\rho\cos\theta\dfrac{\partial P}{\partial y}\\[2ex]\dfrac{\partial P_1}{\partial\rho}=\cos\theta\dfrac{\partial P}{\partial x}+\operatorname{sen}\theta\dfrac{\partial P}{\partial y}\end{cases}$$

Multiplicando-se a primeira equação por $-\operatorname{sen}\theta$, a segunda por $\rho\cos\theta$ e somando membro a membro, obtemos::

$$\frac{\partial P}{\partial x}(x,y)=-\frac{1}{\rho}\operatorname{sen}\theta\frac{\partial P_1}{\partial\theta}(\theta,\rho)+\cos\theta\frac{\partial P_1}{\partial\rho}(\theta,\rho).$$

Procedendo de forma análoga, obtemos:

$$\frac{\partial Q}{\partial y}(x,y)=\frac{1}{\rho}\cos\theta\frac{\partial Q_1}{\partial\theta}(\theta,\rho)+\operatorname{sen}\theta\frac{\partial Q_1}{\partial\rho}(\theta,\rho).$$

b) $\vec{F}(x,y)=P(x,y)\vec{i}+Q(x,y)\vec{j}$

$$\operatorname{div}\vec{F}(x,y)=\frac{\partial P}{\partial x}(x,y)+\frac{\partial Q}{\partial y}(x,y)$$

$$=-\frac{1}{\rho}\operatorname{sen}\theta\frac{\partial P_1}{\partial\theta}(\theta,\rho)+\cos\theta\frac{\partial P_1}{\partial\rho}(\theta,\rho)+\frac{1}{\rho}\cos\theta\frac{\partial Q_1}{\partial\theta}(\theta,\rho)+\operatorname{sen}\theta\frac{\partial Q_1}{\partial\rho}(\theta,\rho).$$

Então

$$\text{div } \bar{F}(x, y) = \text{sen } \theta\left[-\frac{1}{\rho}\frac{\partial P_1}{\partial \theta}(\theta, \rho) + \frac{\partial Q_1}{\partial \rho}(\theta, \rho)\right] + \cos\theta\left[\frac{\partial P_1}{\partial \rho}(\theta, \rho) + \frac{1}{\rho}\frac{\partial Q_1}{\partial \theta}(\theta, \rho)\right].$$

13. *a*) Seja $\varphi(x, y) = f(\frac{x}{y})$, $y > 0$ onde $f(u)$, $u = \frac{x}{y}$, é derivável até 2.ª ordem. Temos

$$\frac{\partial \varphi}{\partial x} = f'(u)\frac{\partial u}{\partial x} = \frac{1}{y}f'(u) \quad \text{e} \quad \frac{\partial^2 \varphi}{\partial x^2} = \frac{1}{y}f''(u)\frac{\partial u}{\partial x} = \frac{1}{y^2}f''(u);$$

$$\frac{\partial \varphi}{\partial y} = f'(u)\frac{\partial u}{\partial y} = \left(-\frac{x}{y^2}\right)f'(u) \quad \text{e} \quad \frac{\partial^2 \varphi}{\partial y^2} = \frac{2x}{y^3}f'(u) + \frac{x^2}{y^4}f''(u).$$

Daí, de $u = \frac{x}{y}$ e de

$$\nabla^2\varphi = \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0, \text{ segue } (1 + u^2)f''(u) + 2uf'(u) = 0.$$

b) Supondo $f'(u) > 0$ temos $\frac{f''(u)}{f'(u)} = -\frac{2u}{1+u^2}$ que é equivalente a

$(\ln f'(u))' = (-\ln(1 + u^2))'$. Daí, $[\ln(1 + u^2)f'(u)]' = 0$

e, portanto, $\ln(1 + u^2)f'(u)) = k$, k constante. Segue que $f'(u) = \frac{A}{1+u^2}$, $A = e^k$. Então, a função $f(u) = A \text{ arctg } u + B$ (A e B constantes) resolve o problema.

Exercícios 1.5

1. Sejam F: $A \subset \mathbb{R}^n \to \mathbb{R}^m$, P um ponto de acumulação de A e $L \in \mathbb{R}^m$

a) $\lim_{x\to P} F(x) = \bar{0}$ ($\bar{0}$ é o vetor nulo de $\mathbb{R}^m$) $\Leftrightarrow \forall \varepsilon > 0, \exists \delta > 0$
tal que, $\forall X \in A$, $0 < \|X - P\| < \delta \Rightarrow \|F(X) - \bar{0}\| < \varepsilon \Leftrightarrow \forall \varepsilon > 0, \exists \delta > 0$ tal que,
$\forall X \in A$, $0 < \|X - P\| < \delta \Rightarrow |\,\|F(x)\| - 0\,| < \varepsilon$

$\Leftrightarrow \lim_{X\to P} \|F(x)\| = 0.$

b) $\lim_{X\to P} F(x) = L \Leftrightarrow \forall \varepsilon > 0, \exists \delta > 0$ tal que, $\forall X \in A$,

$0 < \|X - P\| < \delta \Rightarrow \|F(X) - L\| < \varepsilon.$

Mas $\|F(X) - L\| = |\,\|F(X) - L\| - 0\,|$

Logo,

$\forall \varepsilon > 0, \exists \delta > 0$ tal que, $\forall X \in A$, $0 < \|X - P\| < \delta \Rightarrow |\,\|F(X)\| - L\| - 0\,| < \varepsilon$
$\Leftrightarrow \lim_{X\to P} \|F(X) - L\| = 0$

c) $\lim_{H\to\bar{0}} F(P + H) = L \Leftrightarrow \forall \varepsilon > 0, \exists \delta > 0$ tal que, $\forall H$, com $H + P \in A$,

$0 < \|H - \bar{0}\| < \delta \Rightarrow \|F(P + H) - L\| < \varepsilon$

$\Leftrightarrow \forall \varepsilon > 0, \exists \delta > 0$ tal que, $\forall H$, com $P + H \in A$,

$0 < \|(P + H) - P\| < \delta \Rightarrow \|F(P + H) - L\| < \varepsilon$

$\Leftrightarrow \forall \varepsilon > 0, \exists \delta > 0$ tal que, $\forall X \in A$,

$0 < \|X - P\| < \delta \Rightarrow \|F(X) - L\| < \varepsilon$

$0 < \|X - P\| < \delta \Rightarrow \|F(X) - L\| < \varepsilon$

$\Leftrightarrow \lim_{X\to P} F(X) = L.$

2. Como F é contínua em $G(P) \in B$

$\forall \varepsilon > 0, \exists \delta > 0$ tal que, $\forall Z \in B$, $\|Z - G(P)\| < \delta_1 \Rightarrow \|F(Z) - F(G(P))\| < \varepsilon.$

Como G é contínua em $P \in A$, para o $\delta_1 > 0$ acima, $\exists \delta > 0$ tal que

$\|X - P\| < \delta \Rightarrow \|G(x) - G(P)\| < \delta_1.$

Segue que $\forall \varepsilon > 0$, $\exists \delta > 0$ tal que, $\forall X \in A$,

$0 < \|X - P\| < \delta \Rightarrow \|F(G(x)) - F(G(P))\| < \varepsilon$. Logo,

$H(X) = F(G(X))$ é contínua em P.

3. É dado que existe $M > 0$ tal que, para todo $X \in \Omega$, $\|F(x) - L\| \leq M\|X - P\|$.

Assim,

$\forall \varepsilon > 0, \exists \delta = \frac{\varepsilon}{M}$ tal que, $\forall X \in \Omega$, $0 < \|X - P\| < \frac{\varepsilon}{M} \Rightarrow$

$$\Rightarrow \| F(X) - L \| \leq M \underbrace{\| X - P \|}_{< \frac{\varepsilon}{M}} \Rightarrow \| F(X) - L \| < \varepsilon \Leftrightarrow \lim_{X \to P} F(X) = L.$$

4. Primeiro vamos rever a desigualdade $\| \vec{u} - \vec{v} \| \geq \| \vec{v} \| - \| \vec{u} \|$ que é uma conseqüência da desigualdade triangular. Veja $\| \vec{v} \| = \| -\vec{v} \| = \| (\vec{u} - \vec{v}) + (-\vec{u}) \| \leq \| \vec{u} - \vec{v} \| + \| \vec{u} \|$; daí $\| \vec{u} - \vec{v} \| \geq \| \vec{v} \| - \| \vec{u} \|$.

Vamos então ao exercício.

$$\lim_{X \to P} F(X) = L \Rightarrow \begin{cases} \text{Tomando-se } \varepsilon = \dfrac{\| L \|}{2}, \text{ existe } r > 0 \\ \text{tal que } \forall X \text{ no domínio de } F, \\ 0 < \| X - P \| < r \Rightarrow \| F(x) - L \| < \dfrac{\| L \|}{2}. \end{cases}$$

Tendo em vista a desigualdade acima, $\| F(X) - L \| \geq \| L \| - \| F(X) \|$. Daí e de $\| F(X) - L \| < \dfrac{\| L \|}{2}$, resulta $\| L \| - \| F(X) \| < \dfrac{\| L \|}{2}$ e, portanto, $\| F(x) \| > \dfrac{\| L \|}{2}$.

Assim, existe $r > 0$ tal que

$$0 < \| X - P \| < r \Rightarrow \| F(x) \| > \frac{\| L \|}{2}.$$

CAPÍTULO 2

Exercícios 2.3

1. Sejam $A \subset B \subset \mathbb{R}^2$ e B tem conteúdo nulo. Então, $\forall\, \varepsilon > 0$, existe um número finito de retângulos $A_1, A_2, ..., A_n$ tais que $B \subset A_1 \cup A_2 \cup ... \cup A_n$ e $\sum_{i=1}^{n} m(A_i) < \varepsilon$ onde $m(A_i)$ é a área do retângulo A_i.

(B pode ser coberto por um número finito de retângulo cuja soma das áreas seja tão pequena quanto se queira.)

Como $A \subset B$ então A também pode ser coberto por esses retângulos.

Logo, se $A \subset B$ então $\forall\, \varepsilon < 0$, existe um número finito de retângulos $A_1, A_2, ..., A_n$ tais que $A \subset A_1 \cup A_2 \cup ... \cup A_n$ e $\sum_{i=1}^{n} m(A_i) < \varepsilon$.

Portanto, A tem conteúdo nulo.

2. Seja $\phi \subset \mathbb{R}^2$. Consideremos um conjunto $B \subset \mathbb{R}^2$ tal que B tem conteúdo nulo. Como $\phi \subset B$, então ϕ também terá conteúdo nulo.

3. Suponhamos que o conjunto tenha n pontos. Para cada ponto, considere um quadrado com centro no ponto e com lado estritamente menor que $\sqrt{\dfrac{\varepsilon}{n}}$. Os quadrados cobrem o conjunto e a soma das áreas é menor que ε. Logo, o conjunto tem conteúdo nulo.

CAPÍTULO 3

Exercícios 3.1

1. *a)* $\iint_A f(x, y)\,dx\,dy = \int_0^1 \int_1^2 (x + 2y)\,dx\,dy = \int_0^1 \alpha(y)\,dy$ onde $\alpha(y) = \int_1^2 (x + 2y)\,dx =$

$$= \left[\frac{x^2}{2} + 2xy\right]_1^2 = 2y + \frac{3}{2}.$$

Então, $\iint_A f(x + 2y)\,dx\,dy = \int_0^1 \left[\int_1^2 (x + 2y)\,dx\right] dy =$

$$= \int_0^1 \left(2y + \frac{3}{2}\right) dy = \left[y^2 + \frac{3}{2}y\right]_0^1 = \frac{5}{2}.$$

Invertendo a ordem de integração,

$$\int_1^2 \left[\int_0^1 (x + 2y)\,dy\right] dx = \int_1^2 \left[(xy + y^2)\right]_0^1\,dx =$$

$$= \int_1^2 (x + 1)\,dx = \left[\frac{x^2}{2} + x\right]_1^2 = \frac{5}{2}.$$

c) $\iint_A f(x, y)\,dx\,dy = \int_0^1 \int_1^2 \sqrt{x + y}\,dx\,dy =$

$$= \int_1^2 \left[\frac{2}{3}(x + y)^{3/2}\right]_0^1 dy = \frac{2}{3}\int_0^1 \left[(2 + y)^{3/2} - (1 + y)^{3/2}\right] dy =$$

$$= \frac{2}{3}\cdot\frac{2}{5}\left[(2 + y)^{5/2} - (1 + y)^{5/2}\right]_0^1 = \frac{4}{15}\left[3^{5/2} - 2^{5/2} - 2^{5/2} + 1\right] =$$

$$= \frac{4}{15}(9\sqrt{3} - 8\sqrt{2} + 1).$$

e) $\int_0^1 \int_1^2 dx\,dy = \int_0^1 [x]_1^2\,dy = \int_0^1 dy = [y]_0^1 = 1.$

g) $\int_0^1 \left[\int_1^2 y \cos xy\,dx\right] dy = \int_0^1 [\operatorname{sen} xy]_1^2\,dy =$

$$= \int_0^1 (\operatorname{sen} 2y - \operatorname{sen} y)\,dy = \left[\frac{1}{2}(-\cos 2y) + \cos y\right]_0^1 =$$

$$= -\frac{1}{2}\cos 2 + \cos 1 + \frac{1}{2} - 1 = \cos 1 - \frac{1}{2}[1 + \cos 2].$$

i) $\int_0^1 \left[\int_1^2 y\,e^{xy}\,dx\right] dy = \int_0^1 \left[e^{xy}\right]_1^2\,dy =$

$$= \int_0^1 (e^{2y} - e^y)\,dy = \left[\frac{1}{2}e^{2y} - e^y\right]_0^1 =$$

$$= \frac{1}{2}e^2 - e - \frac{1}{2} + 1 = \frac{1}{2}(1 + e^2) - e.$$

l) $\int_0^1 \left[\int_1^2 x \operatorname{sen} \pi y\,dx\right] dy = \int_0^1 \left[(\operatorname{sen} \pi y)\cdot\frac{x^2}{2}\right]_1^2 dy =$

$$= \int_0^1 \left(2 \operatorname{sen} \pi y - \frac{1}{2}\operatorname{sen} \pi y\right) dy = \frac{3}{2}\int_0^1 \operatorname{sen} \pi y\,dy =$$

$$= \frac{3}{2\pi}[\cos \pi y]_0^1 = \frac{3}{2\pi}[-\cos \pi - (-\cos 0)] = \frac{3}{2\pi}\cdot 2 = \frac{3}{\pi}.$$

2. Temos

$$\iint_A f(x)\,g(y)\,dx\,dy = \int_c^d \int_a^b f(x)\,g(y)\,dx\,dy =$$

$$= \int_c^d \left[\int_a^b f(x)\,g(y)\,dx\right] dy = \int_c^d g(y)\left[\int_a^b f(x)\,dx\right] dy =$$

$$= \left[\int_a^b f(x)\,dx\right]\cdot\left[\int_c^d g(y)\,dy\right].$$

3. *a)* $\iint_A xy^2\,dx\,dy = \int_2^3 \int_1^2 xy^2\,dx\,dy =$

$$= \left(\int_1^2 x\,dx\right)\cdot\left(\int_2^3 y^2\,dy\right) = \left[\frac{x^2}{2}\right]_1^2\cdot\left[\frac{y^3}{3}\right]_2^3 =$$

$$= \left(\frac{3}{2}\right)\cdot\left(\frac{19}{3}\right) = \frac{19}{2}.$$

b) $\iint_A x\cos 2y\,dx\,dy = \left(\int_0^1 x\,dx\right)\cdot\left(\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} \cos 2y\,dy\right) =$

$= \left[\frac{x^2}{2}\right]_0^1 \cdot \left[\frac{1}{2}\operatorname{sen} 2y\right]_{\frac{-\pi}{4}}^{\frac{\pi}{4}} = \left(\frac{1}{2}\right)\cdot\left(\frac{1}{2}+\frac{1}{2}\right) = \frac{1}{2}.$

c) $\iint_A x\ln y\,dx\,dy = \left(\int_0^2 x\,dx\right)\left(\int_1^2 \ln y\,dy\right) =$

$= \left[\frac{x^2}{2}\right]_0^2 \cdot [y\ln y - y]_1^2 = 2(2\ln 2 - 1).$

e) $\iint_A \frac{\operatorname{sen}^2 x}{1+4y^2}\,dx\,dy = \left(\int_0^{\frac{\pi}{2}} \operatorname{sen}^2 x\,dx\right)\left(\int_0^{\frac{1}{2}} \frac{dy}{1+4y^2}\right) =$

$= \left[\frac{1}{2}\operatorname{sen} x\cos x + \frac{x}{2}\right]_0^{\frac{\pi}{2}} \left[\frac{1}{2}\operatorname{arctg} 2y\right]_0^{\frac{1}{2}} =$

$= \left(\frac{\pi}{4}\right)\cdot\left(\frac{\pi}{8}\right) = \frac{\pi^2}{32}.$

(Observe que para todo $n \geqslant 2$, $\int_0^{\frac{\pi}{2}} \operatorname{sen}^n x\,dx = \frac{n-1}{n}\int_0^{\frac{\pi}{2}} \operatorname{sen}^{n-2} dx$,

logo, $\int_0^{\frac{\pi}{2}} \operatorname{sen}^2 x\,dx = \frac{1}{2}\int_0^{\frac{\pi}{2}} dx = \frac{\pi}{4}$.) (Veja Volume 1, Seção 12.3.)

4. *a)* $V = \iint_B (x+2y)\,dx\,dy$, onde B é o retângulo $0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant 1$. Então,

$V = \int_0^1 \left[\int_0^1 (x+2y)\,dx\right] dy = \int_0^1 \left[\frac{x^2}{2} + 2xy\right]_0^1 dy =$

$= \int_0^1 \left(\frac{1}{2}+2y\right) dy = \left[\frac{y}{2} + y^2\right]_0^1 = \frac{3}{2}.$

c) $V = \iint_B xy\,e^{x^2-y^2}\,dx\,dy$, onde B é o retângulo $0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant 1$. Então,

$V = \int_0^1\int_0^1 xye^{x^2-y^2}\,dx\,dy = \int_0^1 xe^{x^2}\,dx \int_0^1 ye^{-y^2}\,dy =$

$= \left[\frac{1}{2}e^{x^2}\right]_0^1 \left[-\frac{1}{2}e^{-y^2}\right]_0^1 = \frac{1}{4}(e-1)(1-e^{-1}).$

e) $V = \iint_B [(x+y+2)-(x+y)]\,dx\,dy$ onde B é o retângulo $1 \leqslant x \leqslant 2, 0 \leqslant y \leqslant 1$.
Então,

$V = \iint_B 2\,dx\,dy = \int_0^1 \left[\int_1^2 2\,dx\right] dy =$

$= \int_0^1 [2x]_1^2\,dy = \int_0^1 2\,dy = [2y]_0^1 = 2.$

f) $V = \iint_B (e^{x+y}-1)\,dx\,dy$, onde B é o retângulo $0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant 1$. Então,

$V = \int_0^1 e^x\,dx \int_0^1 e^y\,dy - \int_0^1 dx \int_0^1 dy$, ou seja,

$V = \left[e^x\right]_0^1 \left[e^y\right]_0^1 - [x]_0^1 [y]_0^1 = (e-1)^2 - 1$ e, portanto,

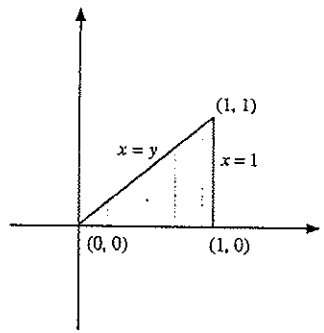
$V = e^2 - 2e.$

5. *a)* $\iint_B y\,dx\,dy$, onde B é o triângulo de vértices (0, 0), (1, 0) e (1, 1).

$\int_0^1 \int_{a(y)}^{b(y)} y\,dx\,dy = \int_0^1 \left[\int_y^1 y\,dx\right] dy =$

$= \int_0^1 [xy]_y^1\,dy = \int_0^1 (y-y^2)\,dy =$

$= \left[\frac{y^2}{2} - \frac{y^3}{3}\right]_0^1 = \frac{1}{6}.$



Invertendo a ordem de integração,

$\iint_B y\,dy\,dx = \int_0^1 \left[\int_{c(x)}^{d(x)} y\,dy\right] dx =$

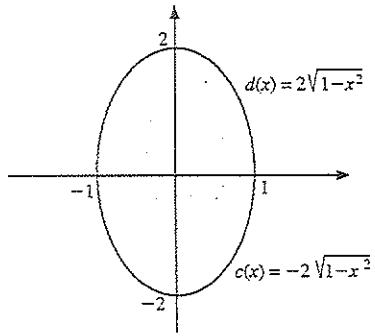
$= \int_0^1 \left[\int_0^x y\,dy\right] dx = \int_0^1 \left[\frac{y^2}{2}\right]_0^x dx = \int_0^1 \frac{x}{2}\,dx =$

$= \left[\frac{x^3}{6}\right]_0^1 = \frac{1}{6}.$

c) Seja $B = \{(x, y) \mid x^2 + 4y^2 \leq 1\}$

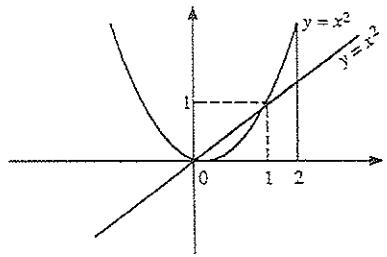
Para cada $x \in [-1, 1]$,

$$\beta(x) = \int_{c(x)}^{d(x)} y\,dy = \int_{-2\sqrt{1-x^2}}^{2\sqrt{1-x^2}} y\,dy = \left[\frac{y^2}{2}\right]_{-2\sqrt{1-x^2}}^{2\sqrt{1-x^2}} = 0$$



Então, $\iint_B y\,dy\,dx = \int_{-1}^{1} \beta(x)\,dx = 0$.

e) $\iint_B y\,dx\,dy$, onde B é a região compreendida entre os gráficos de $y = x$ e $y = x^2$, com $0 \leq x \leq 2$. Para cada $x \in [0, 1]$ temos



$$\iint_{B_1} y\,dy\,dx = \int_0^1 \int_{c(x)}^{d(x)} y\,dy\,dx =$$

$$= \int_0^1 \left[\int_{x^2}^{x} y\,dy\right] dx = \int_0^1 \left[\frac{y^2}{2}\right]_{x^2}^{x} dx = \int_0^1 \left(\frac{x^2}{2} - \frac{x^4}{2}\right) dx =$$

$$= \left[\frac{x^3}{6} - \frac{x^5}{10}\right]_0^1 = \frac{1}{15}.$$

Para cada $x \in [1, 2]$

$$\iint_{B_2} y\,dy\,dx = \int_1^2 \int_{c_1(x)}^{d_1(x)} y\,dy\,dx = \int_1^2 \left[\int_x^{x^2} y\,dy\right] dx =$$

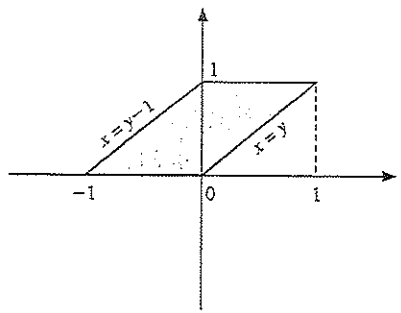
$$= \int_1^2 \left[\frac{y^2}{2}\right]_x^{x^2} dx = \int_1^2 \left(\frac{x^4}{2} - \frac{x^2}{2}\right) dx = \left[\frac{x^5}{10} - \frac{x^3}{6}\right]_1^2 = \frac{29}{15}.$$

Então,

$$\iint_B y\,dy\,dx = \iint_{B_1} y\,dy\,dx + \iint_{B_2} y\,dy\,dx = \frac{1}{15} + \frac{29}{15} = 2.$$

f) $\iint_B y\,dx\,dy$, onde B é o paralelogramo de vértices $(-1, 0)$, $(0, 0)$, $(1, 1)$ e $(0, 1)$.

Para cada y fixo em $[0, 1]$,



$$\iint_B y\,dx\,dy = \int_0^1 \left[\int_{a(y)}^{b(y)} y\,dx\right] dy =$$

$$= \int_0^1 \left[\int_{y-1}^{y} y\,dx\right] dy = \int_0^1 [xy]_{y-1}^{y}\,dy = \int_0^1 y\,dy = \left[\frac{y^2}{2}\right]_0^1 = \frac{1}{2}.$$

Invertendo a ordem de integração,

$$\iint_B y\,dy\,dx = \iint_{B_1} y\,dy\,dx + \iint_{B_2} y\,dy\,dx$$

onde B_1, é o triângulo de vértices $(-1, 0)$, $(0, 0)$ e $(0, 1)$ e B_2 é o triângulo de vértices $(0, 0)$, $(0, 1)$ e $(1, 1)$.

Para cada x fixo em $[-1, 0]$ temos:

$$\int_{-1}^{0}\left[\int_{c(x)}^{d(x)} y\,dy\right]dx = \int_{-1}^{0}\left[\int_{0}^{x+1} y\,dy\right]dx = \int_{-1}^{0}\frac{(x+1)^2}{2}\,dx =$$

$$= \left[\frac{(x+1)^3}{6}\right]_{-1}^{0} = \frac{1}{6}. \quad ①$$

Para cada x fixo em $[0, 1]$ temos

$$\int_0^1\left[\int_{c_1(x)}^{d_1(x)} y\,dy\right]dx = \int_0^1\int_x^1 y\,dy\,dx = \int_0^1\left[\frac{y^2}{2}\right]_x^1 dx =$$

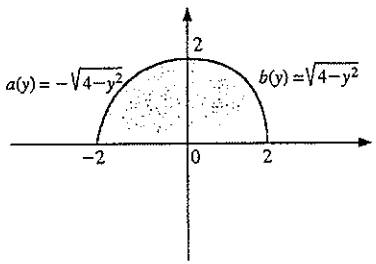
$$= \int_0^1\left(\frac{1}{2} - \frac{x^2}{2}\right)dx = \left[\frac{1}{2}x - \frac{x^3}{6}\right]_0^1 = \frac{1}{2} - \frac{1}{6} = \frac{1}{3} \quad ②$$

Então, de ① e ②,

$$\iint_B y\,dy\,dx = \frac{1}{6} + \frac{1}{3} = \frac{1}{2}.$$

g) $\iint_B y\,dx\,dy$, onde B é o semicírculo $x^2 + y^2 \leq 4$, $y \geq 0$.

Para cada y fixo em $[0, 2]$,



$$\alpha(y) = \int_{a(y)}^{b(y)} y\,dx = \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} y\,dx$$

ou seja,

$$\alpha(y) = [xy]_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} = 2y\sqrt{4-y^2}.$$

Então,

$$\iint_B y\,dx\,dy = \int_0^2 \alpha(y)\,dy = \int_0^2 2y\sqrt{4-y^2}\,dy = \left[-\frac{2}{3}(4-y^2)^{3/2}\right]_0^2 = \frac{16}{3}$$

Invertendo a ordem de integração,

para cada x fixo em $[-2, 2]$,

$$\beta(x) = \int_{c(x)}^{d(x)} y\,dy = \int_0^{\sqrt{4-x^2}} y\,dy = \left[\frac{y^2}{2}\right]_0^{\sqrt{4-x^2}} = 2 - \frac{x^2}{2}$$

Então,

$$\iint_B y\,dy\,dx = \int_{-2}^{2}\beta(x)\,dx = \int_{-2}^{2}\left(2 - \frac{x^2}{2}\right)dx =$$

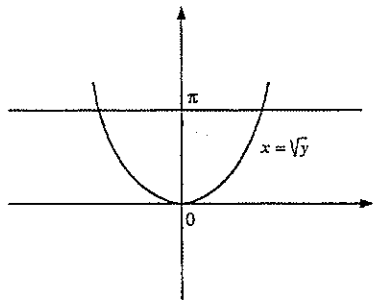
$$= \left[2x - \frac{x^3}{6}\right]_{-2}^{2} = \frac{16}{3}.$$

6. *a)* $\iint_B f(x, y)\,dx\,dy$, onde $f(x, y) = x\cos y$ e $B = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0, x^2 \leq y \leq \pi\}$.

Temos $\alpha(y) = \int_{a(y)}^{b(y)} x\cos y\,dx =$

$$= \int_0^{\sqrt{y}} x\cos y\,dx = \cos y\left[\frac{x^2}{2}\right]_0^{\sqrt{y}}, \text{ ou seja,}$$

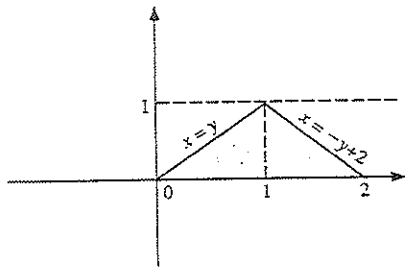
$$\alpha(y) = \frac{y}{2}\cos y.$$



$$\iint_B x \cos y \, dx \, dy = \int_0^{\pi} \alpha(y) \, dy = \int_0^{\pi} \frac{y}{2} \cos y \, dy = \frac{1}{2} \int_0^{\pi} y \cos y \, dy =$$

$$= \frac{1}{2} \left[y \operatorname{sen} y + \cos y \right]_0^{\pi} = \frac{1}{2} (\cos \pi - \cos 0) = -1.$$

c) $\iint_B f(x, y) \, dx \, dy$ onde $f(x, y) = x$ e B é o triângulo de vértices (0, 0), (1, 1) e (2, 0).



$$\iint_B x \, dx \, dy = \int_0^1 \left[\int_{a(y)}^{b(y)} x \, dx \right] dy =$$

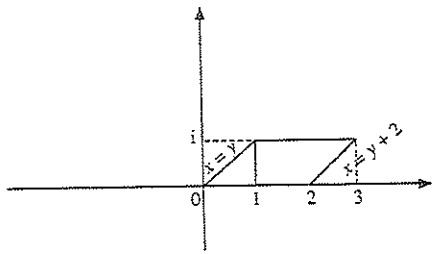
$$= \int_0^1 \left[\int_y^{-y+2} x \, dx \right] dy = \int_0^1 \left[\frac{x^2}{2} \right]_y^{-y+2} dy =$$

$$= \int_0^1 (-2y + 2) \, dy = \left[-y^2 + 2y \right]_0^1 = 1.$$

e) $\iint_B f(x, y) \, dx \, dy$, onde $f(x, y) = x + y$ e B o paralelogramo de vértices (0, 0), (1, 1), (3, 1), (2, 0).

Para cada y fixo em [0, 1],

$$\alpha(y) = \int_{a(y)}^{b(y)} (x + y) \, dx = \int_y^{y+2} (x + y) \, dx = \left[\frac{x^2}{2} + xy \right]_y^{y+2} = 4y + 2.$$



Então,

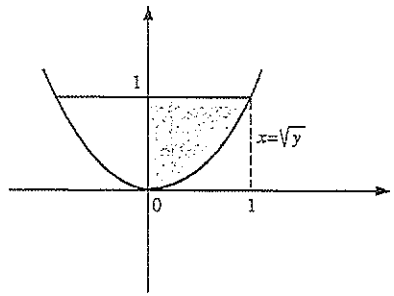
$$\iint_B (x + y) \, dx \, dy = \int_0^1 \left[\int_y^{y+2} (x + y) \, dx \right] dy = \int_0^1 \alpha(y) \, dy =$$

$$= \int_0^1 (4y + 2) \, dy = \left[2y^2 + 2y \right]_0^1 = 4.$$

g) $\iint_B f(x, y) \, dx \, dy$, onde $f(x, y) = xy \cos x^2$ e

$B = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1, x^2 \leq y \leq 1\}$.

Para y em [0, 1],



$$\alpha(y) = \int_{a(y)}^{b(y)} xy \cos x^2 \, dx =$$

$$= \int_0^{\sqrt{y}} xy \cos x^2 \, dx = \left[\frac{y}{2} \operatorname{sen} x^2 \right]_0^{\sqrt{y}}, \text{ ou seja,}$$

$$\alpha(y) = \frac{y}{2} \operatorname{sen} y.$$

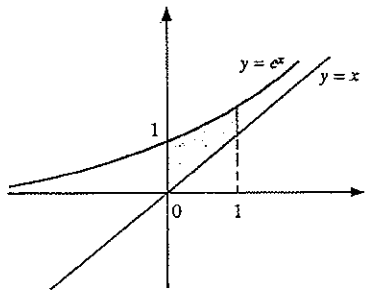
Então,

$$\iint_B f(x, y) \, dx \, dy = \int_0^1 \left[\int_0^{\sqrt{y}} xy \cos x^2 \, dx \right] dy =$$

$$= \int_0^1 \alpha(y) \, dy = \int_0^1 \frac{y}{2} \operatorname{sen} y \, dy = \frac{1}{2} \left[\operatorname{sen} y - y \cos y \right]_0^1 =$$

$$= \frac{1}{2} (\operatorname{sen} 1 - \cos 1).$$

i) $\iint_B f(x, y)\,dx\,dy$, onde $f(x, y) = x + y$ e B é a região compreendida entre os gráficos das funções $y = x$ e $y = e^x$, com $0 \leq x \leq 1$. Temos

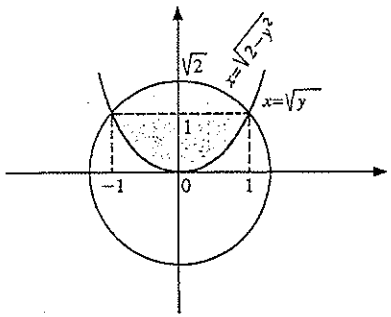


$$\iint_B f(x, y)\,dy\,dx = \int_0^1 \left[\int_{c(x)}^{d(x)} (x + y)\,dy\right] dx = \int_0^1 \left[\int_x^{e^x} (x + y)\,dy\right] dx =$$

$$= \int_0^1 \left[xy + \frac{y^2}{2}\right]_x^{e^x} dx = \int_0^1 \left(xe^x + \frac{e^{2x}}{2} - x^2 - \frac{x^2}{2}\right) dx =$$

$$= \int_0^1 \left(xe^x + \frac{e^{2x}}{2} - \frac{3}{2}x^2\right) dx = \left[xe^x - e^x + \frac{1}{4}e^{2x} - \frac{x^3}{2}\right]_0^1 = \frac{1}{4}(1 + e^2).$$

l) $\iint_B f(x, y)\,dx\,dy$, onde $f(x, y) = x^5 \cos y^3$ e
$B = \{(x, y) \in \mathbb{R}^2 \mid y \geq x^2, x^2 + y^2 \leq 2\}$.



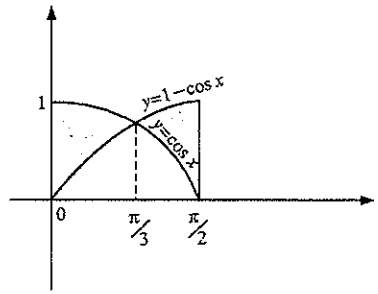
$$\iint_B f(x, y)\,dx\,dy =$$

$$= \int_0^1 \left[\int_{-\sqrt{y}}^{\sqrt{y}} x^5 \cos y^3\,dx\right] dy + \int_1^{\sqrt{2}} \left[\int_{-\sqrt{2-y^2}}^{\sqrt{2-y^2}} x^5 \cos y^3\,dx\right] dy = 0,$$

pois $x^5 \cos y^3$ é uma função ímpar na variável x.

n) $\iint_B f(x, y)\,dx\,dy$, onde $f(x, y) = x$ e B é a região compreendida entre os gráficos de $y = \cos x$ e $y = 1 - \cos x$, com $0 \leq x \leq \frac{\pi}{2}$.

$$\iint_B f(x, y)\,dy\,dx =$$



$$= \int_0^{\frac{\pi}{3}} \left[\int_{c(x)}^{d(x)} x\,dy\right] dx + \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \left[\int_{d(x)}^{c(x)} x\,dy\right] dx =$$

$$= \int_0^{\frac{\pi}{3}} \left[\int_{1-\cos x}^{\cos x} x\,dy\right] dx + \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \left[\int_{\cos x}^{1-\cos x} x\,dy\right] dx =$$

$$= \int_0^{\frac{\pi}{3}} [xy]_{1-\cos x}^{\cos x}\,dx + \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} [xy]_{\cos x}^{1-\cos x}\,dx =$$

$$= \int_0^{\frac{\pi}{3}} (2x\cos x - x)\,dx + \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} (x - 2x\cos x)\,dx =$$

(lembrando que $\int x\cos x\,dx = x\,\text{sen}\,x + \cos x$)

$$= \left[2x\,\text{sen}\,x + 2\cos x - \frac{x^2}{2}\right]_0^{\frac{\pi}{3}} + \left[\frac{x^2}{2} - 2x\,\text{sen}\,x - 2\cos x\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}} =$$

$$= \frac{\pi^2}{72} + \frac{\pi}{3}(2\sqrt{3} - 3).$$

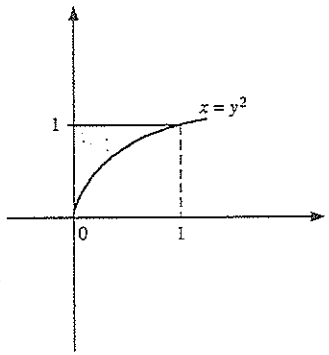
p) $\iint_B f(x, y)\,dx\,dy$, onde $f(x, y) = \sqrt{1+y^3}$ e $B = \{(x, y) \in \mathbb{R}^2 \mid \sqrt{x} \leq y \leq 1\}$.

$$\iint_B f(x, y)\,dx\,dy = \int_0^1 \int_{a(y)}^{b(y)} \left(\sqrt{1+y^3}\right) dx\,dy =$$

$$= \int_0^1 \left[\int_0^{y^2} \left(\sqrt{1+y^3}\right) dx\right] dy =$$

$$= \int_0^1 \left[x\sqrt{1+y^3}\right]_0^{y^2} dy =$$

$$= \int_0^1 y^2\sqrt{1+y^3}\,dy = \frac{2}{9}\left[\left(1+y^3\right)^{3/2}\right]_0^1 = \frac{2}{9}\left(2\sqrt{2}-1\right).$$



r) $\iint_B f(x, y)\,dx\,dy$, onde $f(x, y) = \dfrac{y}{x+y^2}$ e

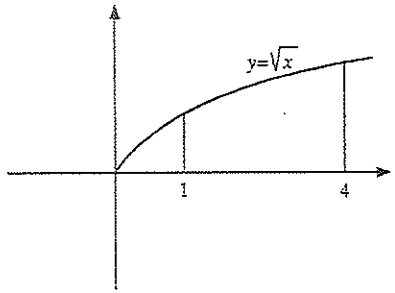
$B = \{(x, y) \in \mathbb{R}^2 \mid 1 \leq x \leq 4 \text{ e } 0 \leq y \leq \sqrt{x}\}$.

Temos

$$\iint_B f(x, y)\,dy\,dx = \int_1^4 \left[\int_0^{\sqrt{x}} \frac{y}{x+y^2}\,dy\right] dx =$$

$$= \int_1^4 \frac{1}{2}\left[\ln\left(x+y^2\right)\right]_0^{\sqrt{x}} dx = \frac{1}{2}\int_1^4 (\ln 2x - \ln x)\,dx = \frac{1}{2}\int_1^4 \ln 2\,dx =$$

$$= \frac{\ln 2}{2}[x]_1^4 = \frac{3}{2}\ln 2.$$



7. *a)* Na integral $\int_0^1 \left[\int_0^x f(x, y)\,dy\right] dx$

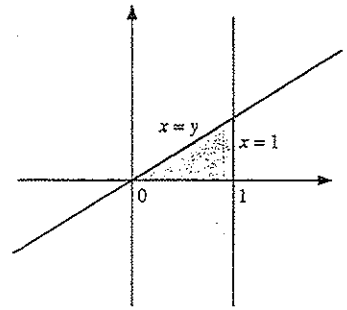
o x está variando no intervalo [0, 1] e, para cada x fixo em [0, 1], y varia de 0 a x. A região de integração é:

$B = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1 \text{ e } 0 \leq y \leq x\}$.

Então,

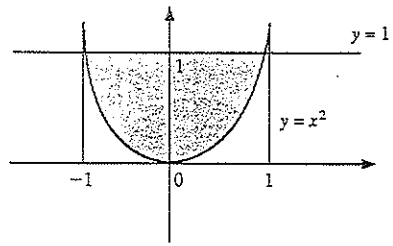
$$\int_0^1 \left[\int_0^x f(x, y)\,dy\right] dx = \int_0^1 \left[\int_y^1 f(x, y)\,dx\right] dy.$$

c) Na integral $\int_0^1 \left[\int_{-\sqrt{y}}^{\sqrt{y}} f(x, y)\,dx\right] dy$



o y está variando no intervalo [0, 1] e, para cada y fixo em [0, 1], x varia de $-\sqrt{y}$ até $\sqrt{y}$. A região de integração é

$B = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq y \leq 1, -\sqrt{y} \leq x \leq \sqrt{y}\}$.



Então, $\int_0^1 \left[\int_{-\sqrt{y}}^{\sqrt{y}} f(x, y)\,dx\right] dy =$

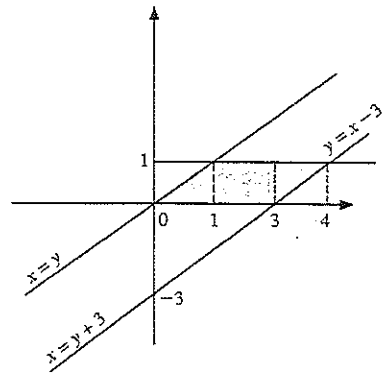
$$= \int_{-1}^1 \left[\int_{x^2}^1 f(x, y)\,dy\right] dx.$$

e) $\int_0^1 \left[\int_y^{y+3} f(x, y)\,dx\right] dy.$

Para cada y fixo em [0, 1], x varia de y até $y + 3$.

A região de integração é:

$B = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq y \leq 1 \text{ e } y \leq x \leq y+3\}$.



Logo,

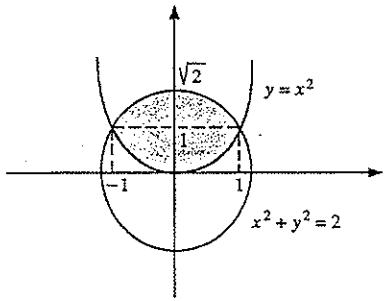
$$\int_0^1\left[\int_y^{y+3} f(x,y)\,dx\right]dy=\int_0^1\left[\int_0^x f(x,y)\,dy\right]dx+$$

$$+\int_1^3\left[\int_0^1 f(x,y)\,dy\right]dx+\int_3^4\left[\int_{x+3}^1 f(x,y)\,dy\right]dx$$

(para cada x fixo em $[0, 1]$, y varia de 0 até x; para cada x fixo em $[1, 3]$, y varia de 0 até 1; para cada x fixo em $[3, 4]$, y varia de $x + 3$ até 1).

g) Na integral $\int_{-1}^1\left[\int_{x^2}^{\sqrt{2-x^2}} f(x,y)\,dy\right]dx$

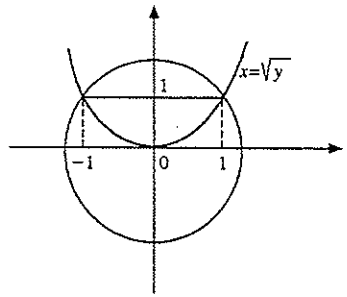
o x está variando em $[-1, 1]$ e, para cada x fixo em $[-1, 1]$, o y varia de x^2 até $\sqrt{2-x^2}$. Seja B a região de integração.

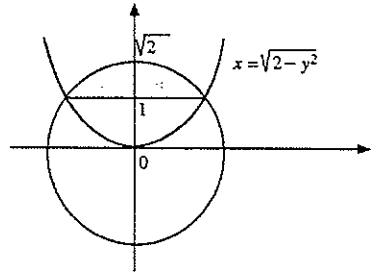


Então B é o conjunto de todos os $(x, y) \in \mathbb{R}^2$ tais que $-1 \leq x \leq 1$, $x^2 \leq y \leq \sqrt{2-x^2}$, ou seja, B é a região do plano compreendida entre os gráficos das funções $y = x^2$ e $y=\sqrt{2-x^2}$, com $-1 \leq x \leq 1$.

Temos: $\int_{-1}^1\left[\int_{x^2}^{\sqrt{2-x^2}} f(x,y)\,dy\right]dx=\iint_{B_1} f(x,y)\,dx\,dy+\iint_{B_2} f(x,y)\,dx\,dy$

onde B_1 é o conjunto de todos os (x, y) tais que $0 \leq y \leq 1$ e $-\sqrt{y} \leq x \leq \sqrt{y}$ e B_2 é o conjunto de todos os (x, y) tais que $1 \leq y \leq \sqrt{2}$ e $-\sqrt{2-y^2} \leq x \leq \sqrt{2-y^2}$





$$\iint_{B_1} f(x,y)\,dx\,dy=\int_0^1\left[\int_{-\sqrt{y}}^{\sqrt{y}} f(x,y)\,dx\right]dy \text{ e}$$

$$\iint_{B_2} f(x,y)\,dx\,dy=\int_1^{\sqrt{2}}\left[\int_{-\sqrt{2-y^2}}^{\sqrt{2-y^2}} f(x,y)\,dx\right]dy$$

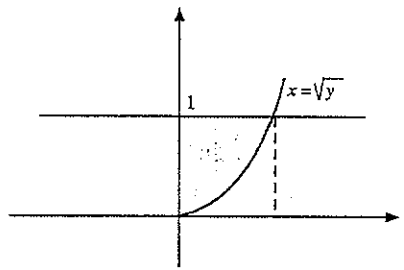
Então,

$$\int_{-1}^1\left[\int_{x^2}^{\sqrt{2-x^2}} f(x,y)\,dy\right]dx=\int_0^1\left[\int_{-\sqrt{y}}^{\sqrt{y}} f(x,y)\,dx\right]dy+\int_1^{\sqrt{2}}\left[\int_{-\sqrt{2-y^2}}^{\sqrt{2-y^2}} f(x,y)\,dx\right]dy.$$

i) Na integral $\int_0^1\left[\int_{x^2}^1 f(x,y)\,dy\right]dx$ o x está variando em $[0, 1]$ e, para cada x fixo em $[0, 1]$, y varia de x^2 até 1.

Assim, $B=\{(x,y)\in\mathbb{R}^2 \mid 0 \leq x \leq 1 \text{ e } x^2 \leq y \leq 1\}$

é a região de integração



Ou ainda, $0 \leq y \leq 1$ e $0 \leq x \leq \sqrt{y}$.

Então,

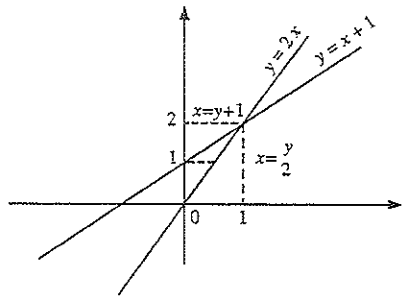
$$\int_0^1 \left[\int_{x^2}^1 f(x, y)\, dy \right] dx =$$

$$= \int_0^1 \left[\int_0^{\sqrt{y}} f(x, y)\, dx \right] dy.$$

l) Na integral $\int_0^1 \left[\int_{2x}^{x+1} f(x, y)\, dy \right] dx$,

a região de integração é

$$B = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1,\ 2x \leq y \leq x + 1\}$$



Então, $0 \leq y \leq 1$ e $0 < x < \frac{y}{2}$

ou $1 \leq y \leq 2$ e $y - 1 \leq x \leq \frac{y}{2}$.

Assim,

$$\int_0^1 \left[\int_{2x}^{x+1} f(x, y)\, dy \right] dx = \int_0^1 \left[\int_0^{\frac{y}{2}} f(x, y)\, dx \right] dy +$$

$$+ \int_1^2 \left[\int_{y-1}^{y/2} f(x, y)\, dx \right] dy.$$

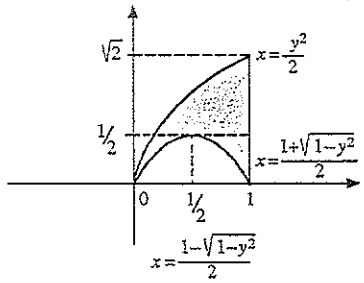
n) Na integral $\int_0^1 \left[\int_{\sqrt{x-x^2}}^{\sqrt{2x}} f(x, y)\, dy \right] dx$

o x está variando em $[0, 1]$ e, para cada x fixo em $[0, 1]$, o y varia de $\sqrt{x - x^2}$ até $\sqrt{2x}$.

Então, B é a região do plano compreendida entre os gráficos das funções $y = \sqrt{2x}$ e $y = \sqrt{x - x^2}$, com $0 \leq x \leq 1$.

Temos $y = \sqrt{2x} \Rightarrow x = \frac{y^2}{2}$

$$y = \sqrt{x - x^2} \Rightarrow -x^2 + x - y^2 = 0 \Rightarrow x = \frac{1 \pm \sqrt{1 - 4y^2}}{2}.$$



Então:

$$\int_{-1}^1 \left[\int_{\sqrt{x-x^2}}^{\sqrt{2x}} f(x, y)\, dy \right] dx =$$

$$= \iint_{B_1} f(x, y)\, dx\, dy + \iint_{B_2} f(x, y)\, dx\, dy + \iint_{B_3} f(x, y)\, dx\, dy$$

onde B_1 é o conjunto de todos os (x, y) tais que $0 \leq y \leq \frac{1}{2}$ e $\frac{y^2}{2} \leq x \leq \frac{1 - \sqrt{1 - 4y^2}}{2}$;

B_2 é o conjunto de todos os (x, y) tais que $0 \leq y \leq \frac{1}{2}$ e $\frac{1 + \sqrt{1 - 4y^2}}{2} \leq x \leq 1$ e B_3 é o conjunto de todos os (x, y) tais que $\frac{1}{2} \leq y \leq \sqrt{2}$ e $\frac{y^2}{2} \leq x \leq 1$.

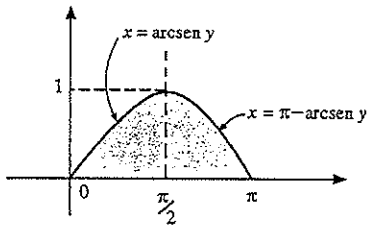
Então,

$$\int_{-1}^{1}\left[\int_{\sqrt{x-x^2}}^{\sqrt{2x}} f(x,y)\,dy\right]dx =$$

$$=\int_{0}^{\frac{1}{2}}\left[\int_{\frac{y^2}{2}}^{\frac{1-\sqrt{1-4y^2}}{2}} f(x,y)\,dx\right]dy+\int_{0}^{\frac{1}{2}}\left[\int_{\frac{1+\sqrt{1-4y^2}}{2}}^{1} f(x,y)\,dx\right]dy +$$

$$+\int_{1/2}^{\sqrt{2}}\left[\int_{\frac{y^2}{2}}^{1} f(x,y)\,dx\right]dy.$$

p) Na integral $\int_0^{\pi}\left[\int_0^{\operatorname{sen} x} f(x,y)\,dy\right]dx$ o x está variando em $[0,\pi]$ e, para cada x fixo em $[0,\pi]$, y varia de 0 até sen x.



A região B de integração é

$$B=\{(x,y)\in\mathbb{R}^2 \mid 0\leq x\leq \pi \text{ e } 0\leq y\leq \operatorname{sen} x\}.$$

Então,

$$\int_0^{\pi}\left[\int_0^{\operatorname{sen} x} f(x,y)\,dy\right]dx=\int_0^{1}\left[\int_{\operatorname{arcsen} y}^{\pi-\operatorname{arcsen} y} f(x,y)\,dx\right]dy.$$

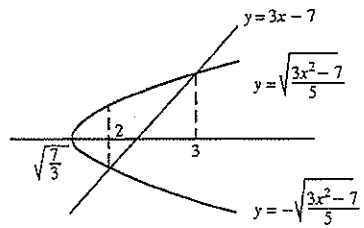
Observe que, para $0\leq x\leq\frac{\pi}{2}$ e $0\leq y\leq 1$, $y=\operatorname{sen} x \Leftrightarrow x=\operatorname{arcsen} y$; por outro lado, como $\frac{\pi}{2}\leq x\leq\pi$ é equivalente a $0\leq\pi-x\leq\frac{\pi}{2}$ e $y=\operatorname{sen} x=\operatorname{sen}(\pi-x)$, segue que, para $\frac{\pi}{2}\leq x\leq\pi$ e $0\leq y\leq 1$,

$$y=\operatorname{sen} x=\operatorname{sen}(\pi-x)\Leftrightarrow \pi-x=\operatorname{arcsen} y\Leftrightarrow x=\pi-\operatorname{arcsen} y.$$

q) $\int_0^{\pi/4}\int_{\operatorname{sen} x}^{\cos x} f(x,y)dydx=\int_0^{\sqrt{2}/2}\int_0^{\operatorname{arcsen} y} f(x,y)dxdy+\int_{\sqrt{2}/2}^{1}\int_0^{\arccos y} f(x,y)dxdy.$

r) $x=\frac{y+7}{3}\Leftrightarrow y=3x-7$ e $x=\sqrt{\frac{7+5y^2}{3}}\Leftrightarrow y=\pm\sqrt{\frac{3x^2-7}{5}}$. Invertendo a ordem de integração, temos

$$\int_{\sqrt{7/3}}^{2}\int_{-\sqrt{(3x^2-7)/5}}^{\sqrt{(3x^2-7)/5}} f(x,y)dydx+\int_2^3\int_{3x-7}^{\sqrt{(3x^2-7)/5}} f(x,y)dydx$$



8. *a)* $V=\iint_A[4-(x+y+2)]\,dx\,dy$, onde A é o círculo $x^2+y^2\leq 1$.

$$V=\int_{-1}^{1}\left[\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}(2-x-y)\,dx\right]dy=\int_{-1}^{1}\left[2x-\frac{x^2}{2}-xy\right]_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}dy=$$

$$=\int_{-1}^{1}\left(4\sqrt{1-y^2}-2y\sqrt{1-y^2}\right)dy=8\int_0^1\sqrt{1-y^2}\,dy.$$

(Observe que $\int_{-1}^{1} y\sqrt{1-y^2}=0$, pois o integrando é função ímpar e $\int_{-1}^{1}\sqrt{1-y^2}\,dy=2\int_0^1\sqrt{1-y^2}\,dy$, pois o integrando é função par.) Fazendo a mudança de variável: $y=\operatorname{sen} u$; $dy=\cos u\,du$, temos

$$V=8\int_0^{\pi/2}\cos^2 u\,du=8\left[\frac{1}{4}\operatorname{sen} 2u+\frac{u}{2}\right]_0^{\pi/2}=8\cdot\frac{\pi}{4}=2\pi.$$

c) $V = \int_{-1}^{1}\left[\int_{0}^{1-x^2}(1-x^2)\,dy\right]dx = \int_{0}^{1}\left[y(1-x^2)\right]_{0}^{1-x^2}\,dx =$

$= \int_{-1}^{1}(1-2x^2+x^4)\,dx = \left[x - \frac{2x^3}{3} + \frac{x^5}{5}\right]_{-1}^{1} = \frac{16}{15}.$

e) $V = 2\int_{-2}^{2}\int_{0}^{\sqrt{1-\frac{x^2}{4}}}\,dy\,dx = 2\int_{-2}^{2}\left[y\right]_{0}^{\sqrt{1-\frac{x^2}{4}}}\,dx =$

$= 2\int_{-2}^{2}\sqrt{1-\frac{x^2}{4}} = 4\int_{0}^{2}\sqrt{1-\frac{x^2}{4}}.$

Fazendo $\frac{x}{2} = \operatorname{sen}\theta$, $dx = 2\cos\theta\,d\theta$.

Temos $x = 0$; $\theta = 0$

$x = 2$; $\theta = \frac{\pi}{2}$.

Então,

$V = 4\int_{0}^{\frac{\pi}{2}}2\cos^2\theta\,d\theta = 8\int_{0}^{\frac{\pi}{2}}\cos^2\theta\,d\theta = 8\left[\frac{1}{2}\theta + \frac{1}{4}\operatorname{sen}2\theta\right]_{0}^{\frac{\pi}{2}} = 8\left[\frac{\pi}{4}\right] = 2\pi.$

g) $V = 2\int_{-a}^{a}\left[\int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}}\sqrt{a^2-y^2}\,dx\right]dy =$

$= 2\int_{-a}^{a}\left[x\sqrt{a^2-y^2}\right]_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}}\,dy =$

$= 4\int_{-a}^{a}(a^2-y^2)\,dy = 8\int_{0}^{a}(a^2-y^2)\,dy =$

$= 8\left[a^2y - \frac{y^3}{3}\right]_{0}^{a} = 8\left(\frac{2a^3}{3}\right) = \frac{16a^3}{3}.$

i) $V = \int_{0}^{1}\left[\int_{0}^{1-x}(1-x-y)\,dy\right]dx = \int_{0}^{1}\left[y - xy - \frac{y^2}{2}\right]_{0}^{1-x}\,dx =$

$= \int_{0}^{1}\left[(1-x) - x(1-x) - \frac{(1-x)^2}{2}\right]dx = \frac{1}{2}\int_{0}^{1}(1-2x+x^2)\,dx =$

$= \frac{1}{2}\left[x - x^2 + \frac{x^3}{3}\right]_{0}^{1} = \frac{1}{2}\cdot\frac{1}{3} = \frac{1}{6}.$

l) Temos $x^2 + y^2 \leq z \leq 2x$.

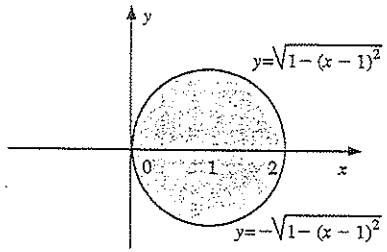
Então, $x^2 + y^2 - 2x \leq 0 \Rightarrow (x^2 - 2x + 1) + y^2 \leq 1 \Rightarrow (x-1)^2 + y^2 \leq 1$

E mais, $0 \leq z \leq 2x - x^2 - y^2$

$V = \int_{0}^{2}\left[\int_{-\sqrt{1-(x-1)^2}}^{\sqrt{1-(x-1)^2}}(2x - x^2 - y^2)\,dy\right]dx =$

$= \int_{0}^{2}\left[2xy - x^2y - \frac{y^3}{3}\right]_{-\sqrt{1-(x-1)^2}}^{\sqrt{1-(x-1)^2}}\,dx =$

$= \int_{0}^{2}\left[4x\sqrt{1-(x-1)^2} - 2x^2\sqrt{1-(x-1)^2} - \frac{2}{3}\left(\sqrt{1-(x-1)^2}\right)^3\right]dx.$



Fazendo $x - 1 = \operatorname{sen}\theta$, temos $dx = \cos\theta\,d\theta$.

$x = 0$; $\theta = -\frac{\pi}{2}$

$x = 2$; $\theta = \frac{\pi}{2}$.

Então,

$V = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left[4(1+\operatorname{sen}\theta)\cos\theta - 2(1+\operatorname{sen}\theta)^2\cos\theta - \frac{2}{3}\cos^2\theta\right]\cos\theta\,d\theta =$

$= 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left[\cos\theta\underbrace{(1-\operatorname{sen}^2\theta)}_{\cos^2\theta} - \frac{\cos^3\theta}{3}\right]\cos\theta\,d\theta = \frac{4}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^4\theta\,d\theta.$

[Utilizando fórmulas de recorrência (veja Vol. 1; Seção 12.9)

$\int\cos^n x\,dx = \frac{1}{n}\cos^{n-1}x\cdot\operatorname{sen}x + \frac{n-1}{n}\int\cos^{n-2}x\,dx.$]

Portanto,

$$V = \frac{4}{3}\left\{\left[\frac{1}{4}\cos^3\theta \operatorname{sen}\theta\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \frac{3}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta\, d\theta\right\}$$

$$V = \frac{4}{3}\cdot\frac{3}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta\, d\theta = \left[\frac{1}{2}\cos x \operatorname{sen} x + \frac{x}{2}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

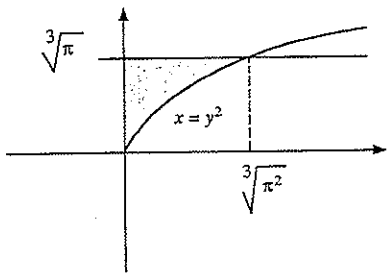
$$V = \left[\frac{1}{4}\operatorname{sen} 2x + \frac{x}{2}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{\pi}{2}.$$

n) $V = \int_0^1\left[\int_0^{1-y}[(3x+y+1)-(4x+2y)]\,dx\right]dy =$

$$= \int_0^1\left[\int_0^{1-y}(-x-y+1)\,dx\right]dy = \int_0^1\left[-\frac{x^2}{2} - xy + x\right]_0^{1-y} dy =$$

$$= \frac{1}{2}\int_0^1 (y^2 - 2y + 1)\,dy = \frac{1}{2}\left[\frac{(y-1)^3}{3}\right]_0^1 = \frac{1}{2}\cdot\frac{1}{3} = \frac{1}{6}.$$

o) $\int_0^{\sqrt[3]{\pi}}\int_0^{y^2} \operatorname{sen} y^3\, dx\, dy = \frac{2}{3}.$



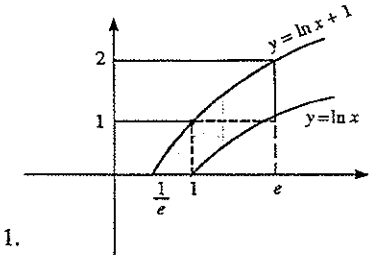
9. *a)* $B = \{(x, y) \in \mathbb{R}^2 \mid \ln x \leq y \leq 1 + \ln x, y \geq 0 \text{ e } x \leq e\}.$

Área $= \int_{\frac{1}{e}}^1 \int_0^{1+\ln x} dy\, dx + \int_1^e \int_{\ln x}^{1+\ln x} dy\, dx$

Área $= \int_{\frac{1}{e}}^1 (1+\ln x)\,dx + \int_1^e dx$

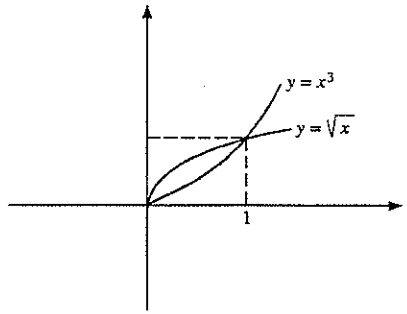
Área $= \int_{\frac{1}{e}}^1 dx + \int_{\frac{1}{e}}^1 \ln x\, dx + \int_1^e dx$

Área $= [x]_{1/e}^1 + [x\ln x - x]_{1/e}^1 + [x]_1^e = e + e^{-1} - 1.$

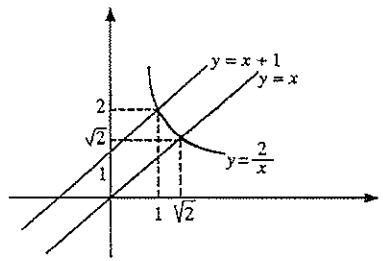


b) $B = \{(x, y) \in \mathbb{R}^2 \mid x^3 \leq y \leq \sqrt{x}\}.$

Área $= \int_0^1 \int_{x^3}^{\sqrt{x}} dy\, dx = \frac{5}{12}.$



c) $B = \{(x, y) \in \mathbb{R}^2 \mid xy \leq 2, x \leq y \leq x+1 \text{ e } x \geq 0\}.$

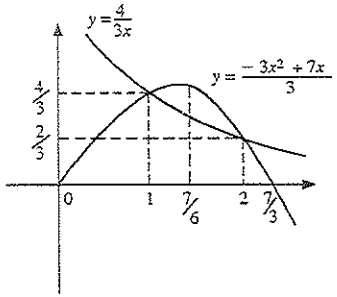


Área $= \int_0^1 \int_x^{x+1} dy\,dx + \int_1^{\sqrt{2}} \int_x^{\frac{2}{x}} dy\,dx = \ln 2 + \frac{1}{2}$.

d) $B = \{(x, y) \in \mathbb{R}^2 \mid x > 0, \frac{4}{x} \leq 3y \leq -3x^2 + 7x\}$.

Temos

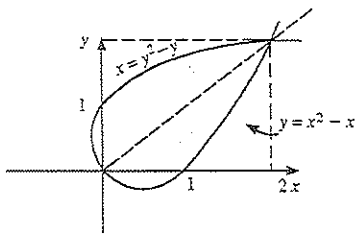
$$\frac{4}{3x} = -\frac{-3x^2 + 7x}{3} \Rightarrow -3x^3 + 7x^2 - 4 = 0$$



Portanto, $x = 1$ e $x = 2$ são abscissas dos pontos de interseção da parábola com a hipérbole.

$$\text{Área} = \int_1^2 \left[\int_{\frac{4}{3x}}^{\frac{-3x^2+7x}{3}} dy \right] dx = \int_1^2 \left[\frac{-3x^2 + 7x}{3} - \frac{4}{3x} \right] dx = \frac{7}{6} - \frac{4}{3} \ln 2.$$

$$\text{Área} = 2 \int_0^2 \left[x - (x^2 - x) \right] dx = \frac{8}{3}.$$



CAPÍTULO 4

Exercícios 4.2

1. *a)* $\iint_B (x^2 + 2y)\,dx\,dy$ onde B é o círculo $x^2 + y^2 \leq 4$.

Façamos a mudança de variável

$$\begin{cases} x = \rho \cos\theta \\ y = \rho \operatorname{sen}\theta \end{cases} \qquad dx\,dy = \left| \frac{\partial(x, y)}{\partial(\rho, \theta)} \right| d\rho\,d\theta.$$

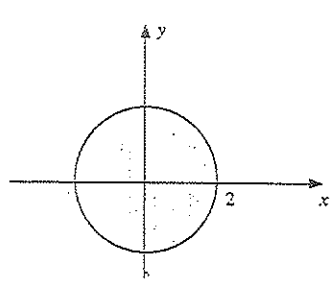
Temos

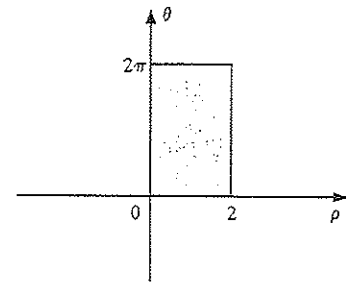
$$\frac{\partial(x, y)}{\partial(\rho, \theta)} = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -\rho \operatorname{sen}\theta \\ \operatorname{sen}\theta & \rho \cos\theta \end{vmatrix} = \rho.$$

Logo,

$$dx\,dy = \rho\,d\rho\,d\theta.$$

Vamos determinar $B_{\rho\theta}$, tal que $B = \varphi(B_{\rho\theta})$, onde φ é a transformação ①.





$B = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 4\}$ $\qquad$ $B_{\rho\theta} = \{(\rho, \theta) \in \mathbb{R}^2 \mid 0 \leq \rho \leq 2 \text{ e } 0 \leq \theta \leq 2\pi\}$

Temos, então,

$$\iint_B (x^2 + 2y)\,dx\,dy = \int_0^{2\pi}\left[\int_0^2 (\rho^2 \cos^2\theta + 2\rho \operatorname{sen}\theta)\,\rho\,d\rho\right]d\theta =$$

$$= \int_0^{2\pi} \cos^2\theta \left[\int_0^2 \rho^3\,d\rho\right]d\theta + \int_0^{2\pi} 2\operatorname{sen}\theta\left[\int_0^2 \rho^2\,d\rho\right]d\theta = 4\pi$$

c) $\iint_B x^2\,dx\,dy$, onde B é o conjunto $4x^2 + y^2 \leq 1$

Façamos a mudança de variável

$$\begin{cases} x = \dfrac{\rho}{2}\cos\theta \\ y = \rho \operatorname{sen}\theta \end{cases} \qquad dx\,dy = \left|\frac{\partial(x, y)}{\partial(\rho, \theta)}\right| d\rho\,d\theta$$

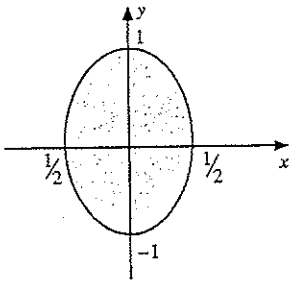
$$\frac{\partial(x, y)}{\partial(\rho, \theta)} = \begin{vmatrix} \dfrac{\partial x}{\partial \rho} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial \rho} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \dfrac{1}{2}\cos\theta & -\dfrac{1}{2}\operatorname{sen}\theta \\ \operatorname{sen}\theta & \rho\cos\theta \end{vmatrix} = \frac{\rho}{2}$$

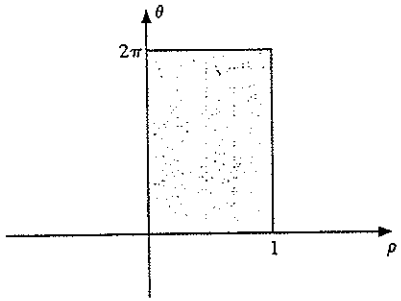
Então,

$$dx\,dy = \frac{\rho}{2}\,d\rho\,d\theta$$

Temos $B = \left\{(x, y) \in \mathbb{R}^2 \;\middle|\; \dfrac{x^2}{\left(\frac{1}{2}\right)^2} + y^2 \leq 1\right\}$

$$B_{\rho\theta} = \{(\rho, \theta) \in \mathbb{R}^2 \mid 0 \leq \rho \leq 1 \text{ e } 0 \leq \theta \leq 2\pi\}$$





Portanto,

$$\iint_B x^2\,dx\,dy = \iint_{B_{\rho\theta}} \left(\frac{1}{2}\rho\cos\theta\right)^2 \frac{\rho}{2}\,d\rho\,d\theta =$$

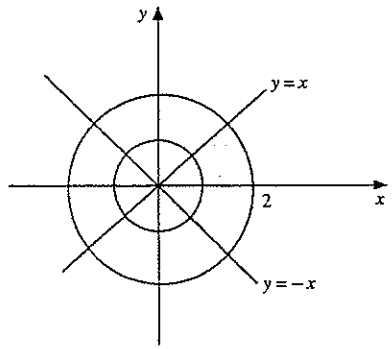
$$= \int_0^{2\pi}\left[\int_0^1 \left(\frac{1}{4}\cos^2\theta\right)\frac{\rho^3}{2}\,d\rho\right]d\theta = \frac{1}{32}\int_0^{2\pi}\cos^2\theta\,d\theta =$$

$$= \frac{1}{32}\left[\frac{1}{2}\cos\theta\operatorname{sen}\theta + \frac{\theta}{2}\right]_0^{2\pi} = \frac{\pi}{32}.$$

e) $\iint_B e^{x^2 + y^2}\,dx\,dy$, onde $B = \{(x, y) \in \mathbb{R}^2 \mid 1 \leq x^2 + y^2 \leq 4, -x \leq y \leq x, x \geq 0\}$

Façamos a mudança de
variável para coordenadas polares:

$$\begin{cases} x = \rho\cos\theta \\ y = \rho\operatorname{sen}\theta \end{cases}$$



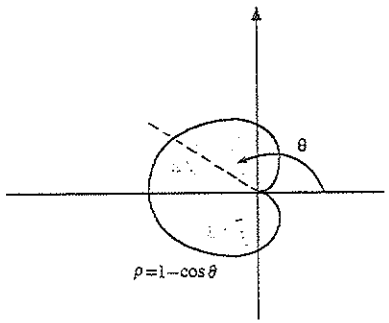
$dx\,dy = \rho\,d\rho\,d\theta$. Temos, ainda, $1 \leq \rho \leq 2$ e $-\dfrac{\pi}{4} \leq \theta \leq \dfrac{\pi}{4}$.

Então,

$$\iint_B e^{x^2 + y^2}\,dx\,dy = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\left[\int_1^2 e^{\rho^2}\,\rho\,d\rho\right]d\theta = \frac{\pi}{4}(e^4 - e).$$

g) $\iint_B x\,dx\,dy$, onde B é o conjunto, no plano xy, limitado pela cardióide $\rho = 1 - \cos\theta$.

Para cada θ fixo em $[0, 2\pi]$, ρ varia de 0 a $1-\cos\theta$.



$$\iint_B x\,dx\,dy = \int_0^{2\pi}\left[\int_0^{1-\cos\theta} \rho^2\cos\theta\,d\rho\right]d\theta =$$

$$= \int_0^{2\pi}\left[\frac{\rho^3}{3}\right]_0^{1-\cos\theta}\cos\theta\,d\theta = \int_0^{2\pi}\frac{(1-\cos\theta)^3}{3}\cos\theta\,d\theta =$$

$$= \int_0^{2\pi}\frac{(1-3\cos\theta+3\cos^2\theta-\cos^3\theta)}{3}\cos\theta\,d\theta =$$

$$= \frac{1}{3}\int_0^{2\pi}\cos\theta\,d\theta - \int_0^{2\pi}\cos^2\theta\,d\theta + \int_0^{2\pi}\cos^3\theta\,d\theta - \frac{1}{3}\int_0^{2\pi}\cos^4\theta\,d\theta =$$

(utilizando a fórmula de recorrência do Vol. 1, Seção 12.9:

$$\int \cos^n\theta\,d\theta = \frac{1}{n}\cos^{n-1}\theta\,\operatorname{sen}\theta + \frac{n-1}{n}\int\cos^{n-2}\theta\,d\theta) =$$

$$= \frac{1}{3}\left[\operatorname{sen}\theta\right]_0^{2\pi} - \left[\frac{1}{2}\cos\theta\,\operatorname{sen}\theta + \frac{\theta}{2}\right]_0^{2\pi} + \left[\frac{1}{3}\cos^2\theta\,\operatorname{sen}\theta + \right.$$

$$\left. + \frac{2}{3}\operatorname{sen}\theta\right]_0^{2\pi} - \left[\frac{1}{4}\cos^3\theta\,\operatorname{sen}\theta + \frac{3}{4}\left(\frac{1}{2}\cos\theta\,\operatorname{sen}\theta + \frac{\theta}{2}\right)\right]_0^{2\pi} =$$

$$= -\pi - \frac{\pi}{4} = -\frac{5\pi}{4}.$$

i) $\iint_B x\,dx\,dy$, onde $B = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 - x \leqslant 0\}$

Temos $x^2 + y^2 - x = 0 \Leftrightarrow x^2 - x + \frac{1}{4} + y^2 = \frac{1}{4} \Leftrightarrow$

$\Leftrightarrow \left(x - \frac{1}{2}\right)^2 + y^2 = \frac{1}{4}$. Então,

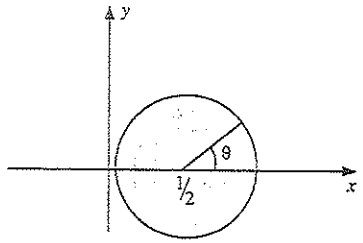
$$B = \{(x, y) \in \mathbb{R}^2 \mid \left(x - \frac{1}{2}\right)^2 + y^2 \leqslant \frac{1}{4}\}.$$

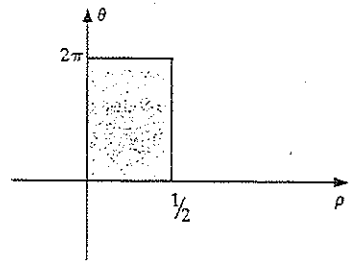
Façamos a mudança de variável

$$\begin{cases} x - \frac{1}{2} = \rho\cos\theta \\ y = \rho\operatorname{sen}\theta \end{cases}$$

$dx\,dy = \rho\,d\rho\,d\theta$. Temos

$B_{\rho\theta} = \{(\rho, \theta) \in \mathbb{R}^2 \mid 0 \leqslant \rho \leqslant \frac{1}{2} \text{ e } 0 \leqslant \theta \leqslant 2\pi\}$.





Então,

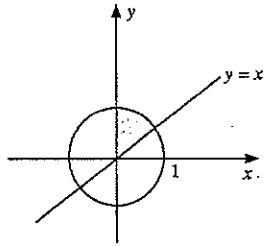
$$\iint_B x\,dx\,dy = \iint_{B_{\rho\theta}}\left(\frac{1}{2} + \rho\cos\theta\right)\rho\,d\rho\,d\theta =$$

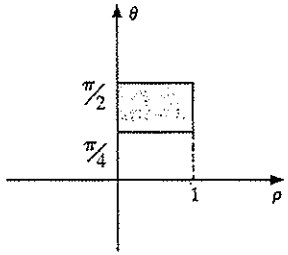
$$= \int_0^{2\pi}\left[\int_0^{\frac{1}{2}}\left(\frac{\rho}{2} + \rho^2\cdot\cos\theta\right)d\rho\right]d\theta = \int_0^{2\pi}\left(\frac{1}{16} + \frac{1}{24}\cos\theta\right)d\theta = \frac{\pi}{8}.$$

l) $\iint_B y^2\,dx\,dy$, onde $B = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leqslant 1, y \geqslant x \text{ e } x \geqslant 0\}$.

$B_{\rho\theta} = \{(\rho, \theta) \in \mathbb{R}^2 \mid 0 \leqslant \rho \leqslant 1 \text{ e } \frac{\pi}{4} \leqslant \theta \leqslant \frac{\pi}{2}\}$

$$\iint_B y^2\,dx\,dy = \iint_{B_{\rho\theta}}\rho^2\operatorname{sen}^2\theta\,\rho\,d\rho\,d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\left[\int_0^1\rho^3\operatorname{sen}^2\theta\,d\rho\right]d\theta = \frac{1}{16}\left[\frac{\pi}{2} + 1\right]$$



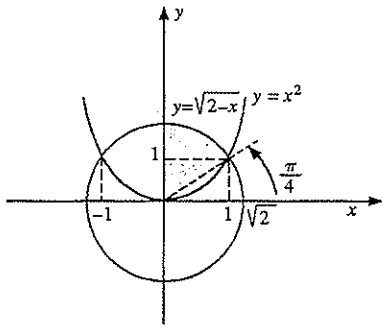


2. *a*) $\int_0^1 \left[\int_{x^2}^{\sqrt{2-x^2}} \sqrt{x^2+y^2}\, dy \right] dx$

Para cada x fixo em [0, 1], y varia de x^2 a $\sqrt{2-x^2}$.

Então, o que se quer é o valor da integral $\iint_B \sqrt{x^2+y^2}\, dx\, dy$, onde

$B = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 2,\ y \geq x^2 \text{ e } 0 \leq x \leq 1\}$.



Façamos a mudança para coordenadas polares

$$\begin{cases} x = \rho \cos\theta \\ y = \rho \operatorname{sen}\theta \end{cases} \qquad dx\, dy = \rho\, d\rho\, d\theta.$$

A equação da parábola $y = x^2$ se escreve em coordenadas polares

$$\rho \operatorname{sen}\theta = (\rho\cos\theta)^2 \Rightarrow \rho = \frac{\operatorname{sen}\theta}{\cos^2\theta}.$$

Então,

$$\iint_B \sqrt{x^2+y^2}\, dx\, dy = \int_0^{\frac{\pi}{4}} \left[\int_0^{\frac{\operatorname{sen}\theta}{\cos^2\theta}} \rho^2\, d\rho \right] d\theta + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left[\int_0^{\sqrt{2}} \rho^2\, d\rho \right] d\theta =$$

$$= \int_0^{\frac{\pi}{4}} \left[\frac{\rho^3}{3} \right]_0^{\frac{\operatorname{sen}\theta}{\cos^2\theta}} d\theta + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left[\frac{\rho^3}{3} \right]_0^{\sqrt{2}} d\theta = \frac{1}{3}\int_0^{\frac{\pi}{4}} \frac{\operatorname{sen}^3\theta}{\cos^6\theta}\, d\theta + \frac{2\sqrt{2}}{3}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} d\theta.$$

Observamos que $\int_0^{\frac{\pi}{4}} \frac{\operatorname{sen}^3\theta}{\cos^6\theta}\, d\theta = \int_0^{\frac{\pi}{4}} \frac{\operatorname{sen}\theta\,(1-\cos^2\theta)}{\cos^6\theta}\, d\theta$

Fazendo $\cos\theta = u$ temos $-\operatorname{sen}\theta\, d\theta = du$.

$\theta = 0;\quad u = 1$

$\theta = \frac{\pi}{4};\ u = \frac{\sqrt{2}}{2}$.

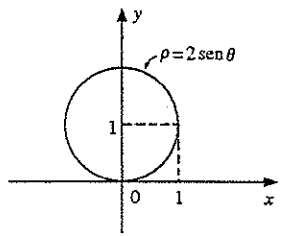
Então $\int_0^{\frac{\pi}{4}} \frac{\operatorname{sen}\theta\,(1-\cos^2\theta)}{\cos^6\theta}\, d\theta = -\int_1^{\frac{\sqrt{2}}{2}} \frac{(1-u^2)}{u^6}\, du = \int_{\frac{\sqrt{2}}{2}}^1 \frac{1-u^2}{u^6}\, du =$

$$= \frac{1}{3}\left[\frac{2}{15}(1+\sqrt{2}) + \frac{\sqrt{2}}{2}\pi \right].$$

c) $\int_0^1 \left[\int_{1-\sqrt{1-x^2}}^{1+\sqrt{1-x^2}} xy\, dy \right] dx.$

Para cada x fixo em [0, 1], y varia de $1-\sqrt{1-x^2}$ a $1+\sqrt{1-x^2}$.
Então, a região de integração é:

$B = \{(x, y) \in \mathbb{R}^2 \mid 1-\sqrt{1-x^2} \leq y \leq 1+\sqrt{1-x^2},\ 0 \leq x \leq 1\}$.



Passando para coordenadas polares

$\begin{cases} x = \rho \cos\theta \\ y = \rho \operatorname{sen}\theta. \end{cases}$

$x^2 + y^2 = 2y \Rightarrow \rho = 2 \operatorname{sen}\theta$

Então,

$$\int_0^1 \left[\int_{1-\sqrt{1-x^2}}^{1+\sqrt{1-x^2}} xy\, dy \right] dx = \int_0^{\frac{\pi}{2}} \left[\int_0^{2\operatorname{sen}\theta} (\rho^2 \operatorname{sen}\theta \cos\theta)\, \rho\, d\rho \right] d\theta =$$

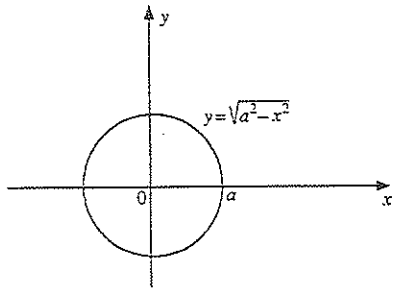
$$= \int_0^{\frac{\pi}{2}} \left[\frac{\rho^4}{4} \right]_0^{2\operatorname{sen}\theta} \operatorname{sen}\theta \cos\theta\, d\theta = 4 \int_0^{\frac{\pi}{2}} \operatorname{sen}^5\theta \cos\theta\, d\theta = \frac{2}{3}$$

e) $\int_0^a \left[\int_0^{\sqrt{a^2-x^2}} \sqrt{a^2 - x^2 - y^2}\, dy \right] dx \quad (a > 0).$

Para cada x fixo em $[0, a]$, y varia de 0 até $\sqrt{a^2 - x^2}$

Temos $B = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq a, 0 \leq y \leq \sqrt{a^2 - x^2}\}$.

$B_{\rho\theta} = \{(\rho, \theta) \in \mathbb{R}^2 \mid 0 \leq \rho \leq a, 0 \leq \theta \leq \frac{\pi}{2}\}$.



Temos:

$$\int_0^a \left[\int_0^{\sqrt{a^2-x^2}} \sqrt{a^2 - x^2 - y^2}\, dy \right] dx =$$

$$= \int_0^{\frac{\pi}{2}} \left[\int_0^a \sqrt{a^2 - \rho^2\cos^2\theta - \rho^2 \operatorname{sen}^2\theta}\ \rho\, d\rho \right] d\theta =$$

$$= \int_0^{\frac{\pi}{2}} \left[\int_0^a \left(-\frac{1}{2}\right)(a^2 - \rho^2)^{\frac{1}{2}}(-2\rho)\, d\rho \right] d\theta =$$

$$= \left(-\frac{1}{2}\right)\frac{2}{3} \int_0^{\frac{\pi}{2}} \left[(a^2 - \rho^2)^{\frac{3}{2}} \right]_0^a d\theta = \frac{a^3\pi}{6}.$$

g) $\iint_B dx\, dy = \int_{-\frac{\pi}{8}}^{\frac{\pi}{4}} \left[\int_0^{\cos 2\theta} \rho\, d\rho \right] d\theta = \frac{1}{2} \int_{-\frac{\pi}{8}}^{\frac{\pi}{4}} \left[\rho^2\right]_0^{\cos 2\theta} d\theta =$

$$= \frac{1}{2} \int_{-\frac{\pi}{8}}^{\frac{\pi}{4}} (\cos 2\theta)^2\, d\theta = \frac{1}{2} \int_{-\frac{\pi}{8}}^{\frac{\pi}{4}} \left(\frac{1}{2} + \frac{1}{2}\cos 4\theta\right) d\theta =$$

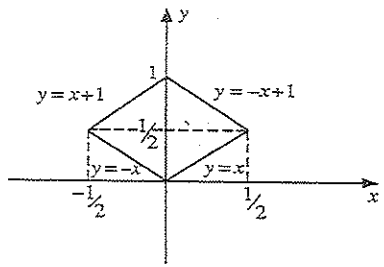
$$= \frac{1}{2}\left[\frac{\theta}{2} + \frac{1}{8} \operatorname{sen} 4\theta \right]_{-\frac{\pi}{8}}^{\frac{\pi}{4}} = \frac{1}{16}\left[\frac{3\pi}{2} + 1\right].$$

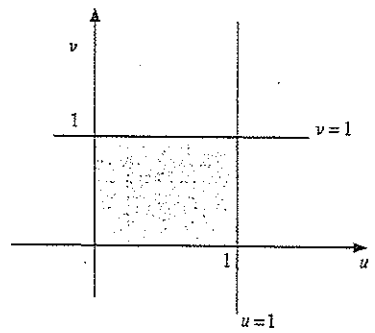
3. $\iint_B \sqrt[3]{y^2 - x^2}\, dx\, dy$, onde B é o paralelogramo de vértices

$(0, 0), \left(\frac{1}{2}, \frac{1}{2}\right), (0, 1)$ e $\left(-\frac{1}{2}, \frac{1}{2}\right)$.

Façamos a mudança de variável

$$\begin{cases} u = y - x \\ v = y + x \end{cases} \Leftrightarrow \begin{cases} x = \frac{v}{2} - \frac{u}{2} \\ y = \frac{v}{2} + \frac{u}{2} \end{cases}$$





De

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} = -\frac{1}{2}$$

segue

$$dx\,dy = \left|\frac{\partial(x, y)}{\partial(u, v)}\right| du\,dv = \frac{1}{2}\,du\,dv.$$

Observamos que a transformação $(u, v) = \psi(x, y)$ dada

por $\begin{cases} u = y - x \\ v = y + x \end{cases}$ é a inversa de $(x, y) = \varphi(u, v)$ dada

por $\begin{cases} x = \frac{v}{2} - \frac{u}{2} \\ y = \frac{v}{2} + \frac{u}{2} \end{cases}$ e que φ é de classe C^1.

Temos que ψ transforma as retas $y = x$, $y = -x + 1$, $y = x + 1$ e $y = -x$, respectivamente, nas retas $u = 0$, $v = 1$, $u = 1$ e $v = 0$.

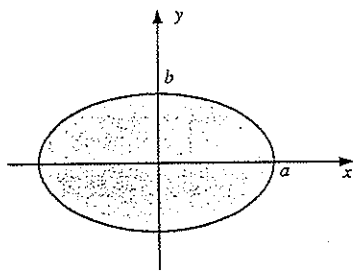
Segue que:

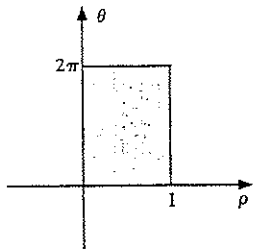
$$\iint_B \sqrt[3]{y^2 - x^2}\,dx\,dy = \int_0^1 \left[\int_0^1 \sqrt[3]{\left(\frac{v}{2} + \frac{u}{2}\right)^2 - \left(\frac{v}{2} - \frac{u}{2}\right)^2}\,\frac{1}{2}\,du\right] dv =$$

$$= \frac{1}{2}\int_0^1 \left[\int_0^1 \sqrt[3]{uv}\,du\right] dv = \frac{9}{32}.$$

4.

$\iint_B dx\,dy$, onde $B = \{(x, y) \in \mathbb{R}^2 \mid \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1, a > 0, b > 0\}$





Façamos a mudança de variável

$$\begin{cases} x = a\,\rho\cos\theta \\ y = b\,\rho\,\text{sen}\,\theta \end{cases}$$

Segue

$$\frac{\partial(x, y)}{\partial(\rho, \theta)} = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} \end{vmatrix} = ab\rho\,(\cos^2\theta + \text{sen}^2\theta).$$ Então,

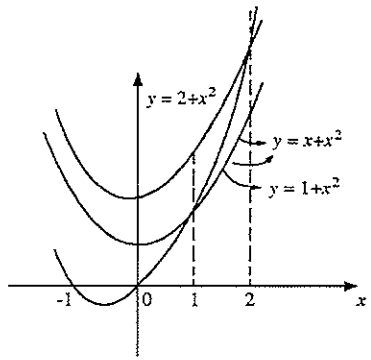
$$dx\,dy = \left|\frac{\partial(x, y)}{\partial(\rho, \theta)}\right| d\rho\,d\theta = ab\rho\,d\rho\,d\theta.$$

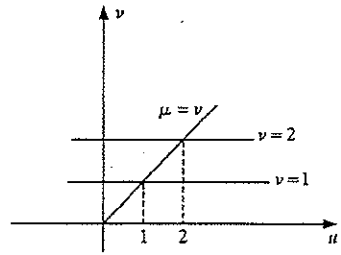
Então, temos

$$\iint_B dx\,dy = \int_0^{2\pi}\left[\int_0^1 ab\rho\,d\rho\right] d\theta = \pi\,ab.$$

5. *a*) Sejam $A = \{(x, y) \in \mathbb{R}^2 \mid 1 + x^2 \leq y \leq 2 + x^2, x \geq 0 \text{ e } y \geq x + x^2\}$ e

$B = \{(u, v) \in \mathbb{R}^2 \mid 1 \geq v \geq 2, v \geq u \text{ e } u \geq 0\}$





Consideremos a transformação $(u, v) = \varphi(x, y)$ dada por

$$\varphi: \begin{cases} u = x \\ v = y - x^2 \end{cases}$$

Observamos que a transformação $(x, y) = \psi(u, v)$ dada por

$$\psi: \begin{cases} x = u \\ y = v - u^2 \end{cases} \qquad \text{é a inversa de } \varphi.$$

Seja $(x, y) \in A$. Então, $x \geq 0$, $y \geq x + x^2$ e $1 + x^2 \leq y \leq 2 + x^2$.

De $x \geq 0 \Rightarrow u \geq 0$

$y \geq x + x^2 \Rightarrow v + u^2 \geq u + u^2 \Rightarrow v \geq u$

$1 + x^2 \leq y \leq 2 + x^2 \Rightarrow 1 + u^2 \leq v + u^2 \leq 2 + u^2 \Rightarrow 1 \leq v \leq 2.$

Portanto, $B = \{(u, v) \mid u \geq 0, v \geq u \text{ e } 1 \leq v \leq 2\} = \varphi(A)$.
Como φ é inversa de ψ, segue então que B é a imagem de A por φ, ou seja, $B = \varphi(A)$.

b) De $\dfrac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} 1 & 0 \\ -2u & 1 \end{vmatrix} = 1$ resulta $dxdy = dudv$. Temos então

Área de $A = \iint_A dxdy = \iint_B dudv$ = Área de B.

Exercícios 4.3

1. *a*) Sejam $\delta(x, y) = y$ e $B = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1, 0 \leq y \leq 1\}$.

O elemento de massa é $dm = \delta(x, y)\, dx\, dy = y\, dx\, dy$.

O centro de massa é o ponto (x_c, y_c) onde

$$x_c = \frac{\iint_B x\, dm}{\iint_B dm} \qquad \text{e} \qquad y_c = \frac{\iint_B y\, dm}{\iint_B dm}$$

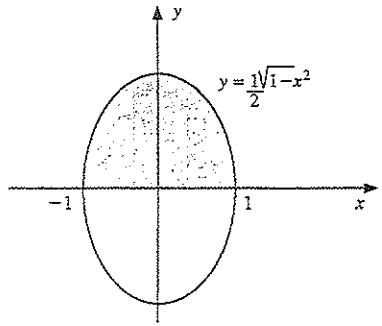
Temos

$$\iint_B dm = \int_0^1 \left[\int_0^1 y\, dx \right] dy = \frac{1}{2},$$

$$\iint_B x\, dm = \int_0^1 \left[\int_0^1 x\, y\, dx \right] dy = \frac{1}{4} \text{ e}$$

$$\iint_B y\, dm = \int_0^1 \left[\int_0^1 y^2\, dx \right] dy = \frac{1}{3}.$$

Portanto, o centro de massa é $\left(\frac{1}{2}, \frac{2}{3}\right)$.



b) $dm = \underbrace{ky}_{\delta(x,y)}\, dx\, dy$

massa de $B = \int_{-1}^{1} \left[\int_0^{\frac{1}{2}\sqrt{1-x^2}} ky\, dy \right] dy =$

$$= \int_{-1}^{1} k \left[\frac{y^2}{2} \right]_0^{\frac{1}{2}\sqrt{1-x^2}} dx = \int_0^1 \frac{1-x^2}{4}\, dx = \frac{k}{6}.$$

$$\iint_B x\, dm = \int_{-1}^{1} \left[\int_0^{\frac{1}{2}\sqrt{1-x^2}} kxy\, dy \right] dx = \frac{k}{8} \int_{-1}^{1} (1 - x^2)\, x\, dx = 0$$

(o integrando é função ímpar).

$$\iint_B y\, dm = \int_{-1}^{1} \left[\int_0^{\frac{1}{2}\sqrt{1-x^2}} ky^2\, dy \right] dx = \frac{k}{24} \int_{-1}^{1} (1 - x^2)^{\frac{3}{2}}\, dx.$$

Fazendo $x = \text{sen } \theta$, $dx = \cos\theta\, d\theta$.

$x = -1;\ \theta = -\dfrac{\pi}{2}$

$x = 1;\quad \theta = \dfrac{\pi}{2}.$

Então,

$$\iint_B y\, dm = \frac{k}{24} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos^2\theta)^{\frac{3}{2}} \cos\theta\, d\theta = \frac{k}{24} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4\theta\, d\theta.$$

Utilizando a fórmula de recorrência

$$\int \cos^n\theta\, d\theta = \frac{1}{n} \cos^{n-1}\theta \,\text{sen}\,\theta + \frac{n-1}{n} \int \cos^{n-2}\theta\, d\theta$$

obtemos

$$\iint_B y\,dm = \frac{k\pi}{64}.$$

Centro de massa: $\left(0, \frac{3\pi}{32}\right)$.

c) $dm = \underbrace{k\sqrt{x^2+y^2}}_{\delta(x,y)}\,dx\,dy.$

B é o triângulo de vértices (0, 0), (1, 0) e (1, 1).
Passando para coordenadas polares

$$\begin{cases} x = \rho\cos\theta \\ y = \rho\,\text{sen}\,\theta \end{cases}$$

$$B_{\rho\theta} = \{(\rho,\theta)\in\mathbb{R}^2 \mid 0 \leqslant \rho \leqslant \sec\theta,\ 0 \leqslant \theta \leqslant \frac{\pi}{4}\}.$$

Massa de $B = \iint_B dm = \iint_B k\overbrace{\sqrt{x^2+y^2}}^{\rho}\ \overbrace{dx\,dy}^{\rho\,d\rho\,d\theta} =$

$$= \int_0^{\frac{\pi}{4}}\left[\int_0^{\sec\theta} k\rho^2\,d\rho\right]d\theta = \frac{k}{3}\int_0^{\frac{\pi}{4}}\sec^3\theta\,d\theta = \frac{k}{6}\left[\sqrt{2}+\ln(1+\sqrt{2})\right].$$

$$\iint_B x\,dm = \int_0^{\frac{\pi}{4}}\left[\int_0^{\sec\theta}(\rho\cos\theta)\,k\rho^2\,d\rho\right]d\theta = \frac{k}{4}\int_0^{\frac{\pi}{4}}\sec^3\theta\,d\theta = \frac{k}{4}\left[\sqrt{2}+\ln(1+\sqrt{2})\right].$$

$$\iint_B y\,dm = \int_0^{\frac{\pi}{4}}\left[\int_0^{\sec\theta}(\rho\,\text{sen}\,\theta)\,k\rho^2\,d\rho\right]d\theta = \frac{k}{4}\int_0^{\frac{\pi}{4}}\text{sen}\,\theta\sec^4\theta\,d\theta =$$

$$= \frac{k}{4}\int_0^{\frac{\pi}{4}}\text{tg}\,\theta\sec^3\theta\,d\theta$$

Fazendo $\begin{cases} u = \sec u,\ du = \sec\theta\,\text{tg}\,\theta\,d\theta. \\ \theta = 0;\ u = \sec 0 = 1 \\ \theta = \frac{\pi}{4};\ u = \sec\frac{\pi}{4} = \sqrt{2} \end{cases}$

Então,

$$\iint_B y\,dm = \frac{k}{4}\int_0^{\frac{\pi}{4}}\sec^2\theta\underbrace{\sec\theta\,\text{tg}\,\theta\,d\theta}_{du} = \frac{k}{4}\int_1^{\sqrt{2}}u^2\,du = \frac{k}{12}(2\sqrt{2}-1).$$

Portanto, as coordenadas do centro de massa são:

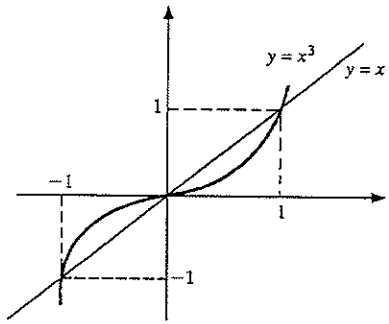
$$x_c = \frac{\iint_B x\,dm}{\iint_B dm} = \frac{\frac{k}{4}\left[\sqrt{2}+\ln(1+\sqrt{2})\right]}{\frac{k}{6}\left[\sqrt{2}+\ln(1+\sqrt{2})\right]} = \frac{3}{2}$$

e

$$y_c = \frac{\iint_B y\,dm}{\iint_B dm} = \frac{\frac{k}{12}(2\sqrt{2}-1)}{\frac{k}{6}\left[\sqrt{2}+\ln(1+\sqrt{2})\right]} = \frac{1}{2}\,\frac{(2\sqrt{2}-1)}{\left[\sqrt{2}+\ln(1+\sqrt{2})\right]}.$$

d) Seja $B = \{(x, y)\in\mathbb{R}^2 \mid x^3 \leqslant y \leqslant x\}$.

Se $\delta(x, y) = 1$ (constante), então a massa de B é igual à área de B.



Massa de $B = 2\int_0^1\left[\int_{x^3}^{x} dy\right]dx = \frac{1}{2}$.

Temos

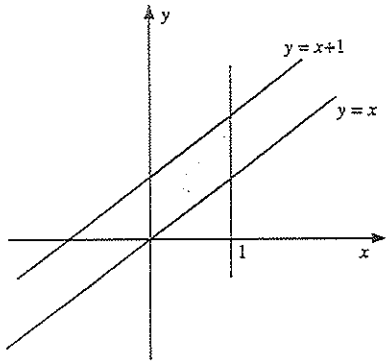
$$\iint_B x\,dm = \iint_B x\,dx\,dy = \int_{-1}^{0}\left[\int_x^{x^3} x\,dy\right]dx + \int_0^1\left[\int_{x^3}^{x} x\,dy\right]dx = 0$$

$$\iint_B y\,dm = \iint_B y\,dx\,dy = \int_{-1}^{0}\left[\int_x^{x^3} y\,dy\right]dx + \int_0^1\left[\int_{x^3}^{x} y\,dy\right]dx = 0$$

Portanto, $(x_c, y_c) = (0, 0)$.

e) Seja $B = \{(x, y)\in\mathbb{R}^2 \mid x \leqslant y \leqslant x+1,\ 0 \leqslant x \leqslant 1\}$

$\delta(x, y) = xy \Rightarrow dm = xy\,dx\,dy$



massa de $B = \iint_B dm = \int_0^1 \left[\int_x^{x+1} xy\, dy \right] dy = \frac{7}{12}$.

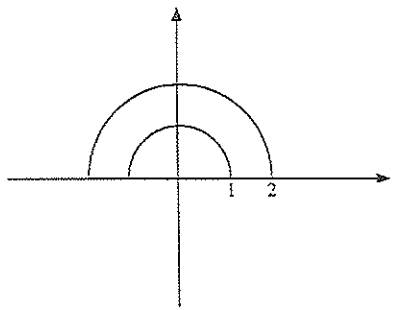
$\iint_B x\, dm = \int_0^1 \left[\int_x^{x+1} x^2 y\, dy \right] dx = \frac{5}{12}$.

$\iint_B y\, dm = \int_0^1 \left[\int_x^{x+1} xy^2\, dy \right] dx = \frac{3}{4}$

Centro de massa: $(x_c, y_c) = \left(\frac{5}{7}, \frac{9}{7} \right)$.

f) Seja $B = \{(x, y) \in \mathbb{R}^2 \mid 1 \leq x^2 + y^2 \leq 4,\ y \geq 0\}$.

Temos $dm = \underbrace{k \sqrt{x^2 + y^2}}_{\delta(x, y)}\, dx\, dy$.



Massa de $B = \iint_B dm$.

Em coordenadas polares:

$dm = k \cdot \rho \cdot \underbrace{\rho\, d\rho\, d\theta}_{dx\, dy} = k\rho^2\, d\rho\, d\theta$.

Massa de $B = \int_0^{\pi} \left[\int_1^2 k\rho^2\, d\rho \right] d\theta = \frac{7k\pi}{3}$

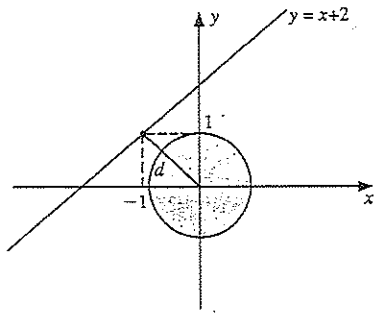
$\iint_B x\, dm = \int_0^{\pi} \left[\int_1^2 k(\rho \cos\theta)\, \rho^2\, d\rho \right] d\theta = 0$

$\iint_B y\, dm = \int_0^{\pi} \left[\int_1^2 k\, (\rho \operatorname{sen}\theta)\, \rho^2\, d\rho \right] d\theta = \frac{15\, k}{2}$

Centro de massa: $(x_c, y_c) = \left(0, \frac{45}{14\pi} \right)$.

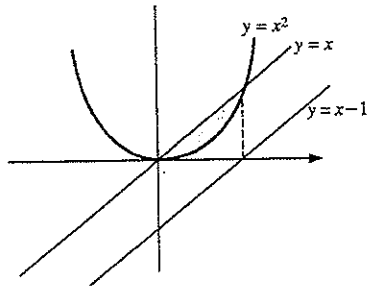
3. *a*) Sejam $B = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$ e a reta $y = x + 2$

Pelo Teorema de Papus, o volume do sólido obtido pela rotação em torno da reta $y = x + 2$ do conjunto $B = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$ é igual ao produto da área de B pelo comprimento da circunferência descrita pelo centro de massa de B.



A área e o centro de massa B são, respectivamente, π e (0, 0). A distância de (0, 0) à reta $y = x + 2$ é $\sqrt{2}$. Logo, o volume é $2\pi^2\sqrt{2}$.

b) Sejam $B = \{(x, y) \in \mathbb{R}^2 \mid x^2 \leq y \leq x\}$ e a reta $y = x - 1$.



Área de $B = \int_0^1 \int_{x^2}^{x} dy\,dx = \int_0^1 (x - x^2)\,dx = \frac{1}{6}$.

Cálculo do centro de massa. Temos

$$\int_0^1 \left[\int_{x^2}^{x} x\,dy\right] dx = \int_0^1 [xy]_{x^2}^{x}\,dx = \int_0^1 (x^2 - x^3)\,dx = \frac{1}{12}$$

$$\int_0^1 \left[\int_{x^2}^{x} y\,dy\right] dx = \int_0^1 \left[\frac{y^2}{2}\right]_{x^2}^{x} dx = \frac{1}{2}\int_0^1 (x^2 - x^4)\,dx = \frac{1}{15};$$

daí, $x_c = \frac{1}{12} \div \frac{1}{6} = \frac{1}{2}$ e $y_c = \frac{1}{15} \div \frac{1}{6} = \frac{2}{5}$, ou seja,

$(x_c, y_c) = \left(\frac{1}{2}, \frac{2}{5}\right)$.

O raio da circunferência descrita pelo centro de massa de B é igual à distância do centro de massa $(x_c, y_c) = \left(\frac{1}{2}, \frac{2}{5}\right)$ à reta $y = x - 1$. Então,

$$r = d = \frac{\left|\frac{2}{5} - \frac{1}{2} + 1\right|}{\sqrt{1+1}} = \frac{9}{10\sqrt{2}} = \frac{9\sqrt{2}}{20}.$$

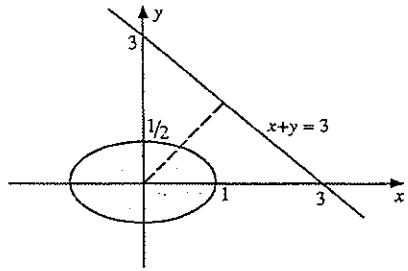
Comprimento da circunferência descrita pelo centro de massa

$$2\pi r = 2\pi \cdot \frac{9\sqrt{2}}{20} = \frac{9\pi\sqrt{2}}{10}.$$

Pelo Teorema de Papus,

$$V = \frac{1}{6} \cdot \frac{9\pi\sqrt{2}}{10} = \frac{3\pi\sqrt{2}}{20}.$$

c) Sejam $B = \{(x, y) \in \mathbb{R}^2 \mid x^2 + 4y \leq 1\}$ e a reta $x + y = 3$.



Façamos a mudança de variável:

$$\begin{cases} x = \rho\cos\theta \\ y = \frac{\rho}{2}\operatorname{sen}\theta. \end{cases}$$

Temos

$$\frac{\partial(x, y)}{\partial(\rho, \theta)} = \begin{vmatrix} \cos\theta & -\rho\operatorname{sen}\theta \\ \frac{1}{2}\operatorname{sen}\theta & \frac{\rho}{2}\cos\theta \end{vmatrix} = \frac{\rho}{2}$$

e

$$dx\,dy = \frac{\rho}{2}\,d\rho\,d\theta.$$

Área de $B = \int_0^{2\pi}\left[\int_0^1 \frac{\rho}{2}\,d\rho\right] d\theta = \frac{1}{4}\int_0^{2\pi} d\theta = \frac{\pi}{2}$.

Evidentemente, $(x_c, y_c) = (0, 0)$. A distância de $(0, 0)$ à reta $x + y = 3$ é $d = \frac{3\sqrt{2}}{2}$; o comprimento da circunferência é $3\pi\sqrt{2}$.

Pelo Teorema de Papus: $V = \frac{\pi}{2} \cdot 3\pi\sqrt{2} = \frac{3\pi^2\sqrt{2}}{2}$.

CAPÍTULO 5

Exercícios 5.4

1. *a)* $\iiint_B xyz\,dx\,dy\,dz$

onde $B = \{(x, y, z) \in \mathbb{R}^3 \mid 0 \leq x \leq 2, 0 \leq y \leq 1 \text{ e } 1 \leq z \leq 2\}$.

$$\iiint_B xyz\,dx\,dy\,dz = \iint_K \left[\int_1^2 xyz\,dz\right] dx\,dy$$

onde $K = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 2, 0 \leq y \leq 1\}$. $\int_1^2 xyz\,dz = xy\left[\frac{z^2}{2}\right]_1^2 = \frac{3xy}{2}$.

Então, temos

$$\iint_K \left[\int_1^2 xyz\,dz\right] dx\,dy = \frac{3}{2}\int_0^1 \left[\int_0^2 xy\,dx\right] dy = \frac{3}{2}.$$

c) $\iiint_B \sqrt{1-z^2}\,dx\,dy\,dz$

onde $B = \{(x, y, z) \in \mathbb{R}^3 \mid 0 \leq x \leq 1, 0 \leq z \leq 1 \text{ e } 0 \leq y \leq z\}$.

$$\iiint_B \sqrt{1-z^2}\,dx\,dy\,dz = \iint_K \left[\int_0^z \sqrt{1-z^2}\,dy\right] dx\,dz$$

onde $K = \{(x, y) \mid 0 \leq x \leq 1, 0 \leq z \leq 1\}$.

$$\iint_K (z\sqrt{1-z^2})\,dx\,dz = \int_0^1 \left[\int_0^1 z\sqrt{1-z^2}\,dx\right] dz$$

$$= \int_0^1 z\sqrt{1-z^2}\,dz = \left(-\frac{1}{2}\right)\frac{2}{3}\left[(1-z^2)^{\frac{3}{2}}\right]_0^1 = \frac{1}{3}.$$

e) $\iiint_B dx\,dy\,dz$ onde $B = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 \leq z \leq 2x\}$.

A projeção no plano xy da interseção do parabolóide $z = x^2 + y^2$ com o plano $z = 2x$ é a circunferência $(x - 1)^2 + y^2 = 1$.

Temos então

$$\iiint_B dx\,dy\,dz = \iint_K \left[\int_{x^2+y^2}^{2x} dz\right] dx\,dy \text{ onde } K \text{ é o círculo } (x-1)^2 + y^2 \leq 1.$$

Façamos $\begin{cases} x - 1 = \rho\cos\theta \\ y = \rho \operatorname{sen}\theta \end{cases} \Rightarrow (x-1)^2 + y^2 = \rho^2$

$0 \leq \theta \leq 2\pi$ e $0 \leq \rho \leq 1$.

$$dx\,dy = \left|\frac{\partial(x, y)}{\partial(\rho, \theta)}\right| d\rho\,d\theta = \rho\,d\rho\,d\theta.$$

Estamos passando para coordenadas polares, com pólo no ponto (1, 0).

$$\iint_K (2x - x^2 - y^2)\,dx\,dy = \iint_K (1 - [(x-1)^2 + y^2])\,dx\,dy$$

$$= \int_0^{2\pi}\int_0^1 (1-\rho^2)\,\rho\,d\rho\,d\theta = \frac{\pi}{2}.$$

g) $\iiint_B dx\,dy\,dz$ onde $B = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 \leq z \leq 2x + 2y - 1\}$.

Vamos determinar a interseção dos gráficos $z = x^2 + y^2$ e $z = 2x + 2y - 1$.

Temos $x^2 + y^2 = 2x + 2y - 1 \Rightarrow x^2 - 2x + 1 + y^2 - 2y + 1 = 1 \Rightarrow$
$\Rightarrow (x - 1)^2 + (y - 1)^2 = 1$.

Façamos $\begin{cases} x - 1 = \rho\cos\theta \\ y - 1 = \rho \operatorname{sen}\theta \end{cases}$ $0 \leq \theta \leq 2\pi$ e $0 \leq \rho \leq 1$. $dx\,dy = \rho\,d\rho\,d\theta$.

Seja $K = \{(x, y) \in \mathbb{R}^2 \mid (x-1)^2 + (y-1)^2 \leq 1\}$. Temos

$$\iiint_B dx\,dy\,dz = \iint_K \left[\int_{x^2+y^2}^{2x+2y-1} dz\right] dx\,dy =$$

$$= \iint_K (1 - [(x-1)^2 + (y-1)^2])\,dx\,dy = \int_0^{2\pi}\left[\int_0^1 (1-\rho^2)\,\rho\,d\rho\right] d\theta = \frac{\pi}{2}.$$

i) $\iiint_B x\,dx\,dy\,dz$

onde $B = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 \leq 4, x \geq 0 \text{ e } x + y \leq z \leq x + y + 1\}$.

$\iint_K \left[\int_{x+y}^{x+y+1} x\,dz\right] dx\,dy$ onde K é o conjunto dos pontos (x, y) tais que $x^2 + y^2 \leq 4, x \geq 0$.

$$\int_{x+y}^{x+y+1} x\,dz = [x\,z]_{x+y}^{x+y+1} = x$$

$$\iint_K x\,dx\,dy = \int_{-2}^2 \left[\int_0^{\sqrt{4-y^2}} x\,dx\right] dy = \int_{-2}^2 \left[\frac{x^2}{2}\right]_0^{\sqrt{4-y^2}} dy =$$

$$= \frac{1}{2}\int_{-2}^{2} \underbrace{(4-y^2)}_{\text{função par}}\, dy = \int_0^2 (4-y^2)\, dy = \left[4y - \frac{y^3}{3}\right]_0^2 = \frac{16}{3}.$$

l) $\iiint_B x\,dx\,dy\,dz$ onde $B = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 - y^2 \leq z \leq 1 - 2y^2\}$.

$x^2 - y^2 = 1 - 2y^2 \Leftrightarrow x^2 + y^2 = 1$. Então,

$\iiint_B x\,dx\,dy\,dz = \iint_K \left[\int_{x^2-y^2}^{1-2y^2} x\,dz\right] dx\,dy$, onde K é o círculo $x^2 + y^2 \leq 1$.

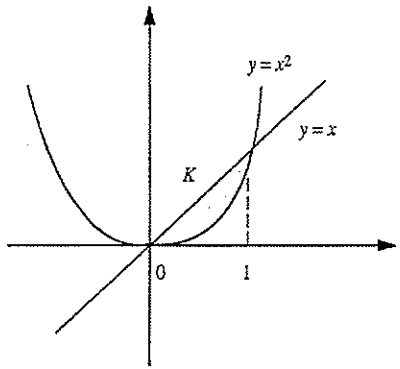
$$\int_{x^2-y^2}^{1-2y^2} x\,dz = x\,[z]_{x^2-y^2}^{1-2y^2} = x(1 - 2y^2 - x^2 + y^2) = x\left[1 - \left(x^2 + y^2\right)\right].$$

$$\iint_K x\left[1 - \left(x^2 + y^2\right)\right] dx\,dy = \int_0^{2\pi}\left[\int_0^1 (\rho\cos\theta)\left(1 - \rho^2\right)\rho\,d\rho\right] d\theta$$

$$= \int_0^{2\pi}\left[\cos\theta \int_0^1 \rho^2\left(1 - \rho^2\right) d\rho\right] d\theta = \int_0^{2\pi}\left(\frac{\rho^3}{3} - \frac{\rho^5}{5}\right)_0^1 \cos\theta\,d\theta$$

$$= \frac{2}{15}\int_0^{2\pi} \cos\theta\,d\theta = 0.$$

n) $\iiint_B x\,dx\,dy\,dz$, onde $B = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 \leq y \leq x,\ 0 \leq z \leq x + y\}$.



$$\iiint_B x\,dx\,dy\,dz = \iint_K \left[\int_0^{x+y} x\,dz\right] dx\,dy =$$

$$= \iint_K (x^2 + xy)\,dx\,dy = \int_0^1 \left[\int_{x^2}^{x} (x^2 + xy)\,dy\right] dx$$

$$= \int_0^1 \left[x^2 y + x\frac{y^2}{2}\right]_{x^2}^{x} dx = \int_0^1 \left(\frac{3}{2}x^3 - x^4 - \frac{x^5}{2}\right) dx =$$

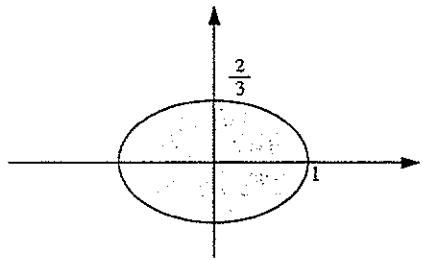
$$= \left[\frac{3}{2}\frac{x^4}{4} - \frac{x^5}{5} - \frac{x^6}{12}\right]_0^1 = \frac{3}{8} - \frac{1}{5} - \frac{1}{12} = \frac{11}{120}.$$

p) $\iiint_B 2z\,dx\,dy\,dz$, onde $B = \{(x, y, z) \in \mathbb{R}^3 \mid 4x^2 + 9y^2 + z^2 \leq 4,\ z \geq 0\}$.

Temos $0 \leq z \leq \sqrt{4 - 4x^2 - 9y^2}$

$$\iint_K \left[\int_0^{\sqrt{4-4x^2-9y^2}} 2z\,dz\right] dx\,dy = \iint_K (4 - 4x^2 - 9y^2)\,dx\,dy \quad \text{onde}$$

$K = \{(x, y) \in \mathbb{R}^2 \mid 4x^2 + 9y^2 \leq 4\}$.



Façamos a mudança de variável:

$$\begin{cases} 2x = \rho\cos\theta \\ 3y = \rho\,\text{sen}\,\theta \end{cases} \quad 0 \leq \rho \leq 2 \text{ e } 0 \leq \theta \leq 2\pi.$$

$$dx\,dy = \left|\frac{\partial(x, y)}{\partial(\rho, \theta)}\right| d\rho\,d\theta = \frac{\rho}{6}\,d\rho\,d\theta.$$

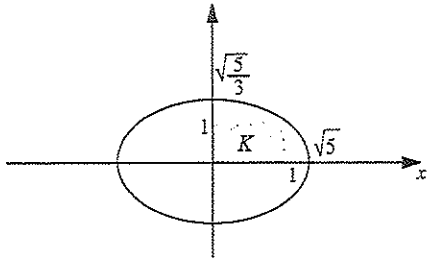
Temos então

$$\iint_K (4 - 4x^2 - 9y^2)\,dx\,dy = \int_0^{2\pi}\int_0^2 (4 - \rho^2)\frac{\rho}{6}\,d\rho\,d\theta$$

$$= \frac{1}{6}\int_0^{2\pi}\left[2\rho^2 - \frac{\rho^4}{4}\right]_0^2 d\theta = \frac{2}{3}\int_0^{2\pi} d\theta = \frac{4\pi}{3}.$$

2. *a)* $B = \{(x, y, z) \in \mathbb{R}^3 \mid 0 \leq x \leq 1,\ 0 \leq y \leq 1 \text{ e } 0 \leq z \leq 5 - x^2 - 3y^2\}$.
A projeção de $z = 5 - x^2 - 3y^2$, $z \geq 0$, no plano xy é a região limitada pela elipse

$5 - x^2 - 3y^2 = 0$.



Então, temos

$$V = \iiint_B dx\,dy\,dz = \iint_K \left[\int_0^{5-x^2-3y^2} dz\right] dy\,dx$$

$$= \iint_K (5 - x^2 - 3y^2)\,dy\,dx, \text{ onde } K = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1, 0 \leq y \leq 1\}.$$

Portanto,

$$V = \int_0^1 \left[\int_0^1 (5 - x^2 - 3y^2)\,dy\right] dx = \int_0^1 \left[5y - x^2y - y^3\right]_0^1 dx$$

$$= \int_0^1 (4 - x^2)\,dx = \left[4x - \frac{x^3}{3}\right]_0^1 = 4 - \frac{1}{3} = \frac{11}{3}.$$

c) $B = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 \leq z \leq 4\}$.

$$V = \iiint_B dx\,dy\,dz = \iint_K \left[\int_{x^2+y^2}^4 dz\right] dy\,dx = \iint_K (4 - x^2 - y^2)\,dx\,dy$$

onde K é o círculo $x^2 + y^2 \leq 4$. Passando para coordenadas polares, temos

$$V = \int_0^{2\pi}\int_0^2 (4 - \rho^2)\,\rho\,d\rho\,d\theta = \int_0^{2\pi}\left[2\rho^2 - \frac{\rho^4}{4}\right]_0^2 d\theta = 4\int_0^{2\pi} d\theta = 8\pi.$$

e) $B = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 \leq 4 \text{ e } x^2 + y^2 + z^2 \leq 9\}$

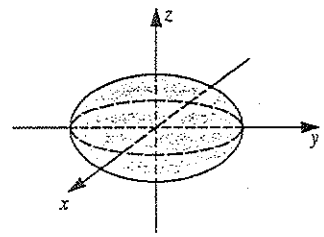
$$\iiint_B dx\,dy\,dz = 2\int_0^{2\pi}\int_0^2 (9 - \rho^2)^{\frac{1}{2}}\,\rho\,d\rho\,d\theta$$

$$= 2\int_0^{2\pi} d\theta\left(-\frac{1}{2}\right)\int_0^2 (9 - \rho^2)^{\frac{1}{2}}\,(-2\rho)\,d\rho$$

$$= \frac{4\pi}{3}\left[-(9 - \rho^2)^{\frac{3}{2}}\right]_0^2 = \frac{4\pi}{3}(-5\sqrt{5} + 27).$$

g) $B = \{(x, y, z) \in \mathbb{R}^3 \mid \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1\,(a > 0, b > 0, c > 0)\}$.

A equação $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ representa um elipsóide de centro na origem.



$$V = 2\iint_K \left[\int_0^c \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}\;dz\right] dx\,dy = 2c\iint_K \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}\;dx\,dy.$$

Façamos $\begin{cases} x = a\,\rho\cos\theta & 0 \leq \theta \leq 2\pi \\ y = b\,\rho\,\text{sen}\,\theta & 0 \leq \rho \leq 1. \end{cases}$

$dx\,dy = ab\rho\,d\rho d\theta$.

Temos então

$$V = 2c\int_0^{2\pi}\left[\int_0^1 (1 - \rho^2)^{\frac{1}{2}}\,ab\rho\,d\rho\right] d\theta = 2abc\int_0^{2\pi} d\theta \cdot \left(-\frac{1}{2}\right)\int_0^1 (1 - \rho^2)^{\frac{1}{2}}\,(-2\rho)\,d\rho$$

$$= 4\,abc\pi \cdot \left(-\frac{1}{3}\right)\left[(1 - \rho^2)^{\frac{3}{2}}\right]_0^1 = 4\,\frac{abc\pi}{3}.$$

h) $B = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 \leq z \leq 4x + 2y\}$

Determinando a projeção no plano xy da interseção de $z = x^2 + y^2$ e $z = 4x + 2y$:

$x^2 + y^2 = 4x + 2y$

$x^2 - 4x + 4 + y^2 - 2y + 1 = 5 \Leftrightarrow (x - 2)^2 + (y - 1)^2 = 5$.

Então

$$V = \iint_K \left[\int_{x^2+y^2}^{4x+2y} dz\right] dx\,dy = \iint_K (4x + 2y - x^2 - y^2)\,dx\,dy$$

onde K é o círculo $(x - 2)^2 + (y - 1)^2 \leq 5$.

Temos $4x + 2y - x^2 - y^2 = 5 - [(x - 2)^2 + (y - 1)^2]$.

Passando para coordenadas polares:

$\begin{cases} x-2=\rho\cos\theta \\ y-1=\rho\operatorname{sen}\theta \end{cases}$

$dx\,dy=\rho\,d\rho\,d\theta$, vem

$$V=\iint_K 5-[(x-2)^2+(y-1)^2]\,dx\,dy=\int_0^{2\pi}\left[\int_0^{\sqrt{5}}(5-\rho^2)\,\rho\,d\rho\right]d\theta=\frac{25\pi}{2}.$$

i) A projeção do sólido $x^2+z^2\leqslant 1$ no plano xy é a faixa $-1\leqslant x\leqslant 1$ e o círculo $x^2+y^2\leqslant 1$ está contido nesta faixa.

Volume $=2\iint_K\left[\int_0^{\sqrt{1-x^2}}dz\right]dx\,dy$, onde K é o círculo $x^2+y^2\leqslant 1$. Passando para polares, vem

$$\text{Volume}=8\int_0^{\frac{\pi}{2}}\int_0^1\sqrt{1-\rho^2\cos^2\theta}\;\rho\,d\rho\,d\theta$$

$$=-\frac{8}{3}\int_0^{\frac{\pi}{2}}\left[\sec^2\theta\,(1-\rho^2\cos^2\theta)^{\frac{3}{2}}\Big|_0^1\right]d\theta$$

$$=-\frac{8}{3}\int_0^{\frac{\pi}{2}}(\sec^2\theta\operatorname{sen}^3\theta-\sec^2\theta)\,d\theta.$$

De $\int\sec^2\theta\,d\theta=\operatorname{tg}\theta$ e $\int\frac{\operatorname{sen}^2\theta}{\cos^2\theta}\underbrace{\operatorname{sen}\theta\,d\theta}_{du}=\cos\theta+\frac{1}{\cos\theta}$ vem

$$\text{Volume}=\frac{8}{3}\lim_{s\to\frac{\pi}{2}^-}\int_0^s(\sec^2\theta-\sec^2\theta\operatorname{sen}^3\theta)\,d\theta, \text{ ou seja,}$$

$$\text{Volume}=\frac{8}{3}\lim_{s\to\frac{\pi}{2}^-}\left[\operatorname{tg}s-\frac{1}{\cos s}-\cos s+2\right]$$

$$=\frac{8}{3}\lim_{s\to\frac{\pi}{2}^-}\left[\frac{\operatorname{sen}s-1}{\cos s}-\cos s+2\right]=\frac{16}{3}.$$

l) $B=\{(x,y,z)\in\mathbb{R}^3\mid (x-a)^2+y^2\leqslant a^2, x^2+y^2+z^2\leqslant 4a^2, z\geqslant 0\ (a>0)\}$.

$$V=\iint_K\int_0^{\sqrt{4a^2-x^2-y^2}}dz\,dx\,dy=\iint_K\sqrt{4a^2-x^2-y^2}\;dx\,dy.$$

Façamos $\begin{cases} x=\rho\cos\theta \\ y=\rho\operatorname{sen}\theta. \end{cases}$

$(x-a)^2+y^2\leqslant a^2\Leftrightarrow x^2+y^2\leqslant 2ax\Leftrightarrow 0\leqslant\rho\leqslant 2a\cos\theta, -\frac{\pi}{2}\leqslant\theta\leqslant\frac{\pi}{2}$.

Então,

$$V=2\int_0^{\frac{\pi}{2}}\left[\int_0^{2a\cos\theta}\sqrt{4a^2-\rho^2}\;\rho\,d\rho\right]d\theta=\left(-\frac{2}{3}\right)\int_0^{\frac{\pi}{2}}\left[(4a^2-\rho^2)^{\frac{3}{2}}\right]_0^{2a\cos\theta}d\theta$$

$$=\frac{2}{3}\int_0^{\frac{\pi}{2}}8a^3\,d\theta-\frac{2}{3}\int_0^{\frac{\pi}{2}}8a^3\operatorname{sen}^3\theta\,d\theta=\frac{16a^3}{3}\left[\frac{\pi}{2}-\frac{2}{3}\right].$$

n) $B=\{(x,y,z)\in\mathbb{R}^3\mid x^2+y^2+z^2\leqslant a^2 \text{ e } z\geqslant\frac{a}{2}\,(a>0)\}$.

$V=\iint_K\int_{\frac{a}{2}}^{\sqrt{a^2-x^2-y^2}}dz\,dx\,dy=\iint_K\left(\sqrt{a^2-x^2-y^2}-\frac{a}{2}\right)dx\,dy$, onde K é o círculo $x^2+y^2\leqslant\frac{3a^2}{4}$.

Em coordenadas polares, K é o conjunto dos pontos (θ,ρ) tais que $0\leqslant\rho\leqslant a\frac{\sqrt{3}}{2}$ e $0\leqslant\theta\leqslant 2\pi$. Então,

$$V=\iint_K\left(\sqrt{a^2-x^2-y^2}-\frac{a}{2}\right)dx\,dy=\int_0^{2\pi}\left[\int_0^{\frac{a\sqrt{3}}{2}}\left(\sqrt{a^2-\rho^2}-\frac{a}{2}\right)\rho\,d\rho\right]d\theta=\frac{5a^3\pi}{24}.$$

p) $B=\{(x,y,z)\in\mathbb{R}^3\mid x^2+2y^2\leqslant z\leqslant 2a^2-x^2\}$.

$$V=\iint_K\int_{x^2+2y^2}^{2a^2-x^2}dz\,dx\,dy=\iint_K\underbrace{(2a^2-x^2-x^2-2y^2)}_{2(a^2-(x^2+y^2))}\,dx\,dy$$

onde K é o círculo $x^2+y^2\leqslant a^2$. (Veja: $x^2+2y^2=2a^2-x^2\Leftrightarrow x^2+y^2=a^2$.)

Passando para coordenadas polares,

$$V=2\int_0^{2\pi}\int_0^a(a^2-\rho^2)\,\rho\,d\rho\,d\theta=\frac{a^4}{2}\cdot 2\pi=a^4\pi.$$

r) $B=\{(x,y,z)\in\mathbb{R}^3\mid 4x^2+9y^2+z^2\leqslant 4 \text{ e } 4x^2+9y^2\leqslant 1\}$.

$$V=\iint_K\left[\int_0^{\sqrt{4-4x^2-9y^2}}dz\right]dx\,dy=\iint_K(4-(4x^2+9y^2))^{\frac{1}{2}}\,dx\,dy,$$

onde K é a região $4x^2+9y^2\leqslant 1$.

Façamos: $\begin{cases} 2x = \tilde{\rho}\cos\theta \\ 3y = \rho \operatorname{sen}\theta \end{cases}$

$0 \leqslant \theta \leqslant 2\pi,\ 0 \leqslant \rho \leqslant 1$ e $dx\,dy = \dfrac{\rho}{6}\,d\rho\,d\theta$.

$$V = \iint_K (4-\rho^2)^{\frac{1}{2}}\,\frac{\rho}{6}\,d\rho\,d\theta = \frac{1}{6}\int_0^{2\pi}\left[\left(-\frac{1}{2}\right)\int_0^1 (4-\rho^2)^{\frac{1}{2}}\,(-2\rho)\,d\rho\right]d\theta$$

$$= -\frac{1}{18}\int_0^{2\pi}\left[(4-\rho^2)^{\frac{3}{2}}\right]_0^1 d\theta = \left(\frac{1}{18}\right)\left[3\sqrt{3}-8\right](2\pi) = \frac{\pi}{9}\left[8-3\sqrt{3}\right].$$

3. Seja $\delta(x, y, z) = x + y + z$.

$$M = \iint_K \left[\int_0^1 (x+y+z)\,dx\right]dy\,dz = \iint_K \left[\frac{x^2}{2} + xy + xz\right]_0^1 dy\,dz$$

onde K é o quadrado $0 \leqslant y \leqslant 1,\ 0 \leqslant z \leqslant 1$.

$$M = \int_0^1\left[\int_0^1\left(\frac{1}{2}+y+z\right)dy\right]dz = \int_0^1\left[\frac{y}{2}+\frac{y^2}{2}+yz\right]_0^1 dz$$

$$= \int_0^1 (1+z)\,dz = \left[z+\frac{z^2}{2}\right]_0^1 = \frac{3}{2}.$$

4. Seja $\delta(x, y, z) = x + y$.

$$M = \iiint_V (x+y)\,dx\,dy\,dz = \iint_A\left[\int_0^{1-x-y}(x+y)\,dz\right]dx\,dy$$

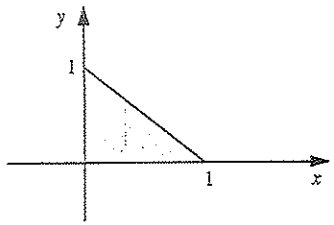
$$= \iint_A (1-x-y)(x+y)\,dx\,dy,\ \text{onde } A \text{ é o triângulo}$$

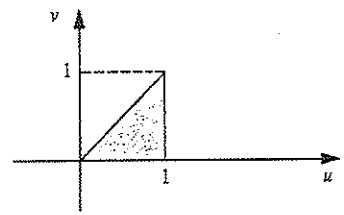
$x + y \leqslant 1,\ x \geqslant 0$ e $y \geqslant 0$

Façamos a mudança de variável:

$$\begin{cases} u = x+y \\ v = x \end{cases} \Leftrightarrow \begin{cases} x = v \\ y = u-v \end{cases} \qquad dx\,dy = \underbrace{\left|\frac{\partial(x,y)}{\partial(u,v)}\right|}_{1}\,du\,dv$$

Esta mudança de variáveis transforma o triângulo de vértices (0, 0), (1, 0) e (0, 1) no triângulo de vértices (0, 0), (1, 0) e (1, 1).





$$M = \int_0^1\int_0^{1-x}(1-x-y)\,(x+y)\,dy\,dx = \int_0^1\int_v^1 (1-u)\,u\,du\,dv$$

$$= \int_0^1\left[\frac{u^2}{2}-\frac{u^3}{3}\right]_v^1 dv = \int_0^1\left(\frac{1}{6}-\frac{v^2}{2}+\frac{v^3}{3}\right)dv = \left[\frac{1}{6}v-\frac{v^3}{6}+\frac{v^4}{12}\right]_0^1 = \frac{1}{12}.$$

5. Seja $\delta(x, y, z) = 2z$.

$$M = \iint_A\left[\int_c^2 2z\,dz\right]dx\,dy = \iint_A\left[z^2\right]_0^2\,dx\,dy$$

$$= \iint_A 4\,dx\,dy \text{ onde A é o círculo } x^2 + y^2 \leqslant 4.$$

Portanto, temos (passando para coordenadas polares):

$$M = \iint_A\int_0^2 4\,\rho\,d\rho\,d\theta = \int_0^{2\pi}\left[2r^2\right]_0^2 d\theta = 8\int_0^{2\pi}d\theta = 16\,\pi.$$

6. Seja $\delta(x, y, z) = k(x^2 + y^2)$, onde k é a constante de proporcionalidade.

$$M = \iint_A\left[\int_{\sqrt{x^2+y^2}}^1 k(x^2+y^2)\,dz\right]dx\,dy = k\iint_A\left[(x^2+y^2)\,z\right]_{\sqrt{x^2+y^2}}^1 dx\,dy$$

$$= k\iint_A (x^2+y^2)\,(1-\sqrt{x^2+y^2})\,dx\,dy,\ \text{onde } A \text{ é o círculo } x^2 + y^2 \leqslant 1.$$

Passando para coordenadas polares,

$$M = \int_0^{2\pi}\int_0^1 \rho^2(1-\rho)\,\rho\,d\rho\,d\theta = k\int_0^{2\pi}\left[\frac{\rho^4}{4}-\frac{\rho^5}{5}\right]_0^1 d\theta = \frac{k}{20}\int_0^{2\pi}d\theta = \frac{k\pi}{10}.$$

Exercícios 5.5

1. *a)* $\iiint_B x\,dx\,dy\,dz$ onde $B = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \leqslant 4,\ x \geqslant 0\}$.

Passando para coordenadas esféricas:

$$\begin{cases} x = \rho\operatorname{sen}\varphi\cos\theta \\ y = \rho\operatorname{sen}\varphi\operatorname{sen}\theta \\ z = \rho\cos\varphi \end{cases}$$

$$\frac{\partial(x, y, z)}{\partial(\theta, \rho, \varphi)} = \rho^2 \operatorname{sen} \varphi \quad \Rightarrow \quad dx\,dy\,dz = \rho^2 \operatorname{sen} \varphi \, d\theta \, d\rho \, d\varphi$$

com $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}, 0 \leq \varphi \leq \pi, 0 \leq \rho \leq 2$.

$$\iiint_B x\,dx\,dy\,dz = \iiint_{B_{\theta\varphi\rho}} \rho \operatorname{sen} \varphi \cos\theta \cdot \rho^2 \operatorname{sen} \varphi \, d\rho \, d\theta \, d\varphi$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\theta \, d\theta \int_0^{\pi} \operatorname{sen}^2 \varphi \, d\varphi \int_0^2 \rho^3 \, d\rho = 4\pi.$$

c) $\iiint_B x\,dx\,dy\,dz$ onde $B = \{(x, y, z) \in \mathbb{R}^3 \mid \frac{x^2}{4} + \frac{y^2}{9} + z^2 \leq 1, x \geq 0\}$

Passando para coordenadas esféricas,

$$\begin{cases} x = 2\rho \operatorname{sen} \varphi \cos\theta \\ y = 3\rho \operatorname{sen} \varphi \operatorname{sen} \theta \\ z = \rho \cos\varphi \end{cases}$$

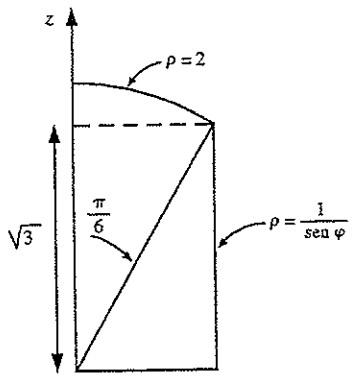
com $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}, 0 \leq \varphi \leq \pi, 0 \leq \rho \leq 1$

$dx\,dy\,dz = 6\rho^2 \operatorname{sen} \varphi \, d\varphi \, d\rho \, d\theta.$

$$\iiint_B x\,dx\,dy\,dz = \iiint_{B_{\theta\varphi\rho}} 2\rho \operatorname{sen} \varphi \cos\theta \, 6\rho^2 \operatorname{sen} \varphi \, d\varphi \, d\rho \, d\theta$$

$$= 12 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\theta \, d\theta \int_0^1 \rho^3 \, d\rho \int_0^{\pi} \operatorname{sen}^2 \varphi \, d\varphi = 3\pi.$$

f) $\iiint \sqrt{x^2 + y^2 + z^2}\, dx\,dy\,dz$, onde B é a interseção da semi-esfera $x^2 + y^2 + z^2 \leq 4, z \geq 0$ com o cilindro $x^2 + y^2 \leq 1$.



O sólido é obtido pela rotação em torno do eixo z da figura acima.

Façamos a interseção da semi-esfera com o cilindro

$$\begin{cases} x^2 + y^2 + z^2 = 4 \\ x^2 + y^2 = 1. \end{cases} \Rightarrow z^2 = 3$$

Passando para coordenadas esféricas:

$$\begin{cases} x = \rho \cos\theta \operatorname{sen} \varphi \\ y = \rho \operatorname{sen} \theta \operatorname{sen} \varphi \\ z = \rho \cos\varphi \end{cases}$$

$dx\,dy\,dz = \rho^2 \operatorname{sen} \varphi \, d\varphi \, d\rho \, d\theta$. Temos

$\cos\varphi = \frac{\sqrt{3}}{2}$ e, portanto, $\varphi = \frac{\pi}{6}$;

$x^2 + y^2 = 1 \Leftrightarrow \rho^2 \operatorname{sen}^2 \varphi = 1 \Leftrightarrow \rho = \frac{1}{\operatorname{sen} \varphi}$, que é a equação da superfície cilíndrica $x^2 + y^2 = 1$ em coordenadas esféricas.

Temos então

$$\iiint \sqrt{x^2 + y^2 + z^2}\, dx\,dy\,dz = \int_0^{2\pi} \int_0^{\frac{\pi}{6}} \int_0^2 \rho^3 \operatorname{sen} \varphi \, d\rho \, d\varphi \, d\theta$$

$$+ \int_0^{2\pi} \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \int_0^{\frac{1}{\operatorname{sen} \varphi}} \rho^3 \operatorname{sen} \varphi \, d\rho \, d\varphi \, d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^{\frac{\pi}{6}} \operatorname{sen} \varphi \, d\varphi \int_0^2 \rho^3 \, d\rho + \int_0^{2\pi} \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left[\int_0^{\frac{1}{\operatorname{sen} \varphi}} \rho^3 \, d\rho \right] \operatorname{sen} \varphi \, d\varphi \, d\theta$$

$$= 2\pi \left[-\cos\frac{\pi}{6} + \cos 0 \right] \left[\frac{\rho^4}{4} \right]_0^2 + 2\pi \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left[\frac{\rho^4}{4} \right]_0^{\frac{1}{\operatorname{sen} \varphi}} \operatorname{sen} \varphi \, d\varphi$$

$$= 8\pi \left(-\frac{\sqrt{3}}{2} + 1 \right) + \frac{\pi}{2} \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \operatorname{cosec}^3 \varphi \, d\varphi$$

$$= -4\sqrt{3}\,\pi + 8\pi + \frac{\pi}{2} \left[-\frac{1}{2} \operatorname{cosec} \varphi \cot \varphi + \frac{1}{2} \ln(\operatorname{cosec} \varphi - \cot \varphi) \right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$= -4\sqrt{3}\,\pi + 8\pi + \frac{\pi}{2} \left[\sqrt{3} + \frac{1}{2} \ln(\sqrt{3} + 2) \right]$$

$$= -\frac{7\sqrt{3}\pi}{2} + 8\pi + \frac{\pi}{4} \ln(\sqrt{3} + 2).$$

(Utilizamos a fórmula de recorrência:

$$\int \operatorname{cosec}^n \varphi \, d\varphi = -\frac{1}{n-1} \operatorname{cosec}^{n-2} \varphi \cot \varphi + \frac{n-2}{n-1} \int \operatorname{cosec}^{n-2} \varphi \, d\varphi$$

veja Exemplo 7, Seção 12.10, Volume I, pág. 394.)

2. Seja o elipsóide $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leqslant 1$.

Passando para coordenadas esféricas:

$$\begin{cases} x = a\rho \operatorname{sen} \varphi \cos \theta \\ y = b\rho \operatorname{sen} \varphi \operatorname{sen} \theta \quad \text{com } 0 \leqslant \theta \leqslant 2\pi \text{ e } 0 \leqslant \varphi \leqslant \pi. \\ z = c\rho \cos \varphi \end{cases}$$

$$dx\,dy\,dz = \left|\frac{\partial(x, y, z)}{\partial(\rho, \theta, \varphi)}\right| = abc\rho^2 \operatorname{sen} \varphi \, d\rho \, d\varphi \, d\theta.$$

De $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leqslant 1$ segue que $0 \leqslant \rho \leqslant 1$.

Temos $B_{\theta\varphi\rho} = \{(\theta, \varphi, \rho) \mid 0 \leqslant \theta \leqslant 2\pi, 0 \leqslant \varphi \leqslant \pi, 0 \leqslant \rho \leqslant 1\}$

Então,

$$V = \iiint_B dx\,dy\,dz = \iiint_{B_{\theta\varphi\rho}} abc\rho^2 \operatorname{sen} \varphi \, d\rho \, d\varphi \, d\theta$$

$$= abc \int_0^{2\pi} d\theta \int_0^{\pi} \operatorname{sen} \varphi \, d\varphi \int_0^1 \rho^2 \, d\rho = abc \, [\theta]_0^{2\pi} \cdot [-\cos \varphi]_0^{\pi} \cdot \left[\frac{\rho^3}{3}\right]_0^1 = \frac{4\,abc\pi}{3}.$$

4. Seja o conjunto $B = \{(x, y, z) \in \mathbb{R}^3 \mid z \geqslant \sqrt{x^2 + y^2} \text{ e } x^2 + y^2 + z^2 \leqslant 2az \, (a > 0)\}$.

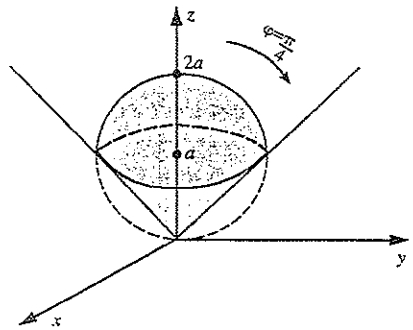
Da equação da esfera $x^2 + y^2 + z^2 = 2az$ segue que
$x^2 + y^2 + z^2 - 2az + a^2 = a^2$, ou seja, $x^2 + y^2 + (z - a)^2 = a^2$.

Portanto, a esfera tem centro em $(0, 0, a)$ e raio igual a a.

Em coordenadas esféricas, a equação da esfera $x^2 + y^2 + z^2 = 2az$ é $\rho = 2a \cos \varphi$ e a equação do cone $z = \sqrt{x^2 + y^2}$ é $z = \rho \operatorname{sen} \varphi$. De $\rho \cos \varphi = \rho \operatorname{sen} \varphi$ segue $\varphi = \frac{\pi}{4}$.

Portanto a região de integração (em coordenadas esféricas) é

$$B_{\theta\rho\varphi} = \left\{(\theta, \rho, \varphi) \mid 0 \leqslant \theta \leqslant 2\pi, 0 \leqslant \rho \leqslant 2a \cos \varphi, 0 \leqslant \varphi \leqslant \frac{\pi}{4}\right\}$$



Temos então

$$V = \iiint_B dx\,dy\,dz = \iiint_{B_{\theta\rho\varphi}} \rho^2 \operatorname{sen} \varphi \, d\rho \, d\varphi \, d\theta$$

$$= \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \left[\int_0^{2a\cos\varphi} \rho^2 \operatorname{sen} \varphi \, d\rho\right] \operatorname{sen} \varphi \, d\varphi \, d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^{\frac{\pi}{4}} \left[\frac{\rho^3}{3}\right]_0^{2a\cos\varphi} \operatorname{sen} \varphi \, d\varphi = 2\pi \int_0^{\frac{\pi}{4}} \frac{8a^3}{3} \cos^3 \varphi \operatorname{sen} \varphi \, d\varphi = \pi a^3.$$

Exercícios 5.7

1.

$$I = \iiint_B r^2 \, dm, \text{ onde } r = \sqrt{x^2 + y^2} \text{ e } dm = \underbrace{\delta(x, y, z)}_{k} dx\,dy\,dz.$$

$$I = \iiint_B (x^2 + y^2) \, k \, dx\,dy\,dz = k \iint_A \left[\int_0^{4-x-y} (x^2 + y^2) \, dz\right] dy\,dx$$

$$= k \iint_A (x^2 + y^2)(4 - x - y) \, dx\,dy \text{ onde } A \text{ é o triângulo}$$

$x + y \leqslant 4, x \geqslant 0, y \geqslant 0.$

$$I = k \int_0^4 \left[\int_0^{4-x} (4x^2 + 4y^2 - x^3 - xy^2 - x^2y - y^3) \, dy\right] dx$$

$$= k \int_0^4 \left[4x^2y + \frac{4y^3}{3} - x^3y - \frac{xy^3}{3} - \frac{x^2y^2}{2} - \frac{y^4}{4}\right]_0^{4-x} dx$$

$$= k \int_0^4 \left[4x^2(4-x) + \frac{4(4-x)^3}{3} - x^3(4-x) - \frac{x(4-x)^3}{3} - \frac{x^2(4-x)^2}{2} - \frac{(4-x)^4}{4}\right] dx =$$

$$= \frac{k}{12}\int_0^4 (7x^4 - 64x^3 + 192x^2 - 256x + 256)\,dx = \frac{512}{15}\,k.$$

2. Vamos considerar o cubo $0 \leq x \leq L$, $0 \leq y \leq L$ e $0 \leq z \leq L$ e calcular o momento de inércia em relação ao eixo z. Temos $\delta(x, y, z) = k$ (cubo homogêneo),

$r(x, y, z) = \sqrt{x^2 + y^2}$, $dm = \delta(x, y, z)\,dx\,dy\,dz = k\,dx\,dy\,dz$.

Então,

$$I = \iiint_C r^2\,dm = \iiint_C (x^2 + y^2)\,k\,dx\,dy\,dz$$

$$= k\iint_A \left[\int_0^L (x^2 + y^2)\,dz\right] dx\,dy = kL\int_0^L \left[\int_0^L (x^2 + y^2)\,dx\right] dy$$

$$= \frac{2}{3}kL^5 = \frac{2}{3}L^2\underbrace{(kL^3)}_{M}.$$

Então, $I = \frac{2}{3}L^2M$, onde M é a massa do cubo.

4. Seja o cilindro homogêneo $(x - a)^2 + y^2 \leq a^2$ e $0 \leq z \leq h$.

a) $I = \iiint_B r^2\,dm$

onde $r = r(x, y, z)$ é a distância do ponto (x, y, z) à reta $x = a$ e $y = 0$. Então, $r^2 = (x - a)^2 + y^2$ e $dm = \underbrace{(x, y, z)}_{k}\,dx\,dy\,dz$ (cilindro homogêneo).

Em coordenadas cilíndricas, façamos:

$$\begin{cases} x - a = \rho\cos\theta \\ y = \rho\,\text{sen}\,\theta \\ z = z \end{cases}$$

com $(\rho, \theta, z) \in B_{\theta\rho\varphi} = \{(\rho, \theta, z) \mid 0 \leq \rho \leq a, 0 \leq \theta \leq 2\pi, 0 \leq z \leq h\}$.

Então, temos

$$I = \iiint_B r^2\,dm = \iiint_B r^2\,k\,dx\,dy\,dz = \iiint_{B_{\rho\theta z}} \rho^2 k\rho\,d\rho\,d\theta\,dz$$

$$= \int_0^{2\pi} d\theta \int_0^a \rho^3\,d\rho \int_0^h dz = 2\pi\left(\frac{a^4}{4}\right)h = \frac{k\pi a^4 h}{2} = \frac{a^2}{2}\underbrace{(k\pi a^2 h)}_{M}.$$

Então, $I = \frac{a^2M}{2}$, onde M é a massa do cilindro.

b) $I = \iiint_B (x^2 + y^2)\,dm$, onde $dm = \underbrace{\delta(x, y, z)}_{k}\,dx\,dy\,dz$.

Em coordenadas cilíndricas (tomando o pólo na origem),

$$\begin{cases} x = \rho\cos\theta \\ y = \rho\,\text{sen}\,\theta \\ z = z \end{cases}$$

$dx\,dy\,dz = \rho\,d\rho\,d\theta\,dz$.

Temos,

$(x - a)^2 + y^2 = a^2 \iff \rho = 2a\cos\theta$.

Então $B_{\rho\theta z}$ é dado por $0 \leq \rho \leq 2a\cos\theta$, $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, $0 \leq z \leq h$.

$$I = \iiint_B (x^2 + y^2)\,k\,dx\,dy\,dz = k\iiint_{B_{\rho\theta z}} \rho^2 \cdot \rho\,d\rho\,d\theta\,dz$$

$$= k\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2a\cos\theta}\left[\int_0^h \rho^3\,dz\right] d\rho\,d\theta = kh\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2a\cos\theta} \rho^3\,d\rho\,d\theta$$

$$= kh\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\frac{\rho^4}{4}\right]_0^{2a\cos\theta} d\theta = 8a^4\,hk\int_0^{\frac{\pi}{2}} \cos^4\theta\,d\theta = \frac{3a^2}{2}\underbrace{(\pi a^2\,hk)}_{M}.$$

Então, $I = \frac{3Ma^2}{2}$, onde $M = k\pi a^2 h$ é a massa do cilindro.

6. Temos

$I_{CM} = \frac{Ma^2}{2}$ é o momento de inércia em relação à reta que passa pelo centro de massa $(x = a)$ (Exercício 4 item *a*, desta seção)

$I = \frac{3Ma^2}{2}$ é o momento de inércia em relação a um eixo paralelo (eixo 0z), a uma distância $h = a$ (Exercício 4 item *b*, desta seção).

Portanto, pelo teorema de Steiner,

$$\underbrace{\frac{Ma^2}{2}}_{I_{cm}} + Mh^2 = \frac{Ma^2}{2} + Ma^2 = 3\underbrace{\frac{Ma^2}{2}}_{I}.$$

(Pelo teorema de Steiner, $I = I_{cm} + Mh^2$.)

7. Centro de massa da semi-esfera homogênea $x^2 + y^2 + z^2 \leq R^2$ e $z \geq 0$ $(R > 0)$.

Precisamos calcular apenas z_c tendo em vista a simetria do corpo em relação ao eixo 0z.

Temos

$$z_c = \frac{\iiint_B xk\, dx\, dy\, dz}{\iiint_B k\, dx\, dy\, dz} = \frac{\iiint z\, dx\, dy\, dz}{\iiint dx\, dy\, dz}$$

Passando para coordenadas cilíndricas: $\begin{cases} x = \rho\cos\theta \\ y = \rho\,\text{sen}\,\theta \\ z = z \end{cases}$

$x^2 + y^2 = \rho^2 \Rightarrow z^2 \leq R^2 - \rho^2$

$$\iiint_B dx\, dy\, dz = \int_0^{2\pi}\int_0^R\int_0^{\sqrt{R^2-\rho^2}} \rho\, dz\, d\rho\, d\theta$$

$$= \int_0^{2\pi}\int_0^R (R^2-\rho^2)^{\frac{1}{2}}\,\rho\, d\rho\, d\theta = \frac{2\pi R^3}{3}.$$

$$\iiint z\, dx\, dy\, dz = \int_0^{2\pi}\int_0^R\int_0^{\sqrt{R^2-\rho^2}} \rho\, z\, dz\, d\rho\, d\theta$$

$$= \int_0^{2\pi}\left[\int_0^R \frac{(R^2-\rho^2)}{2}\rho\, d\rho\right] d\theta = \frac{\pi R^4}{4}.$$

Então, $z_c = \frac{\pi R^4}{4}\cdot\frac{3}{2\pi R^3} = \frac{3R}{8}$.

Centro de massa da semi-esfera homogênea $\left(0, 0, \frac{3R}{8}\right)$

Por coordenadas esféricas,

$$\iiint dx\, dy\, dz = \left(\frac{1}{2}\right)\int_0^{2\pi}\int_0^{\pi}\int_0^R \rho^2\,\text{sen}\,\varphi\, d\rho\, d\varphi\, d\theta$$

$$= \left(\frac{1}{2}\right)\left[\int_0^R \rho^2\, d\rho\right]\cdot\left[\int_0^{\pi}\text{sen}\,\varphi\, d\varphi\right]\cdot\left[\int_0^{2\pi} d\theta\right]$$

$$= \left(\frac{1}{2}\right)\cdot\left[\frac{\rho^3}{3}\right]\left[-\cos\varphi\right]_0^{\pi}\,[\theta]_0^{2\pi} = \left(\frac{1}{2}\right)\cdot\left(\frac{R^3}{3}\right)\cdot 2\cdot 2\pi = \frac{2\pi R^3}{3}$$

$$\iiint z\, dx\, dy\, dz = \int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^R (\rho\cos\varphi)\,\rho^2\,\text{sen}\,\varphi\, d\varphi\, d\rho\, d\theta$$

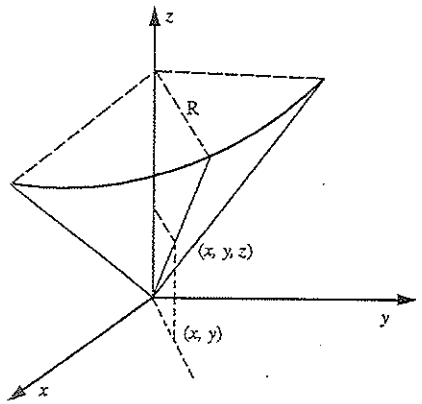
$$= \left[\int_0^{2\pi} d\theta\right]\cdot\left[\int_0^R \rho^3\, d\rho\right]\cdot\left[\int_0^{\frac{\pi}{2}}\text{sen}\,\varphi\cos\varphi\, d\varphi\right]$$

$$= [\theta]_0^{2\pi}\cdot\left[\frac{\rho^4}{4}\right]_0^R\cdot\left[\frac{\text{sen}^2\,\varphi}{2}\right]_0^{\frac{\pi}{2}} = 2\pi\cdot\frac{R^4}{4}\cdot\frac{1}{2} = \frac{\pi R^4}{4}$$

$z_c = \frac{3R}{8}$. Centro de massa: $\left(0, 0, \frac{3k}{8}\right)$.

9. *a*) O centro de massa está no eixo z tendo em vista a simetria do corpo.
Então, $x_c = y_c = 0$ e

$z_c = \dfrac{\iiint_B zk\, dx\, dy\, dz}{\iiint_B k\, dx\, dy\, dz}$, onde B é o cone.



A equação do cone obtém-se por semelhança de triângulos:

$$\frac{z}{h} = \frac{\sqrt{x^2+y^2}}{R}$$

Portanto, B é o cone $\frac{h}{R}\sqrt{x^2+y^2} \leq z \leq h$.

Em coordenadas cilíndricas,

$$M = \iiint_B dx\, dy\, dz = \iiint_{B_{\rho\theta z}} \rho\, d\rho\, d\theta\, dz =$$

$$= \int_0^{2\pi}\int_0^R \left[\int_{\frac{\rho h}{R}}^{h} \rho\, dz\right] d\rho\, d\theta = \int_0^{2\pi}\left[\int_0^R \left(h - \frac{\rho h}{R}\right)\rho\, d\rho\right] d\theta$$

$$= \int_0^{2\pi}\left[\frac{h\rho^2}{2} - \frac{h}{R}\frac{\rho^3}{3}\right]_0^R d\theta = \int_0^{2\pi}\left(h\frac{R^2}{2} - \frac{hR^2}{3}\right) d\theta$$

$$= \frac{hR^2}{3}\int_0^{2\pi} d\theta = \frac{\pi R^2 h}{3}.$$

$$\iiint_B z\, dx\, dy\, dz = \iiint_{B_{\rho\theta z}} z\rho\, dz\, d\rho\, d\theta = \int_0^{2\pi}\int_0^R \left[\int_{\frac{\rho h}{R}}^{h} z\, dz\right]\rho\, d\rho\, d\theta$$

$$= \frac{h^2R^2\pi}{4}.$$

Portanto, $z_c = \frac{\pi R^2 h^2}{4} \div \frac{\pi R^2 h}{3} = \frac{3}{4}h$.

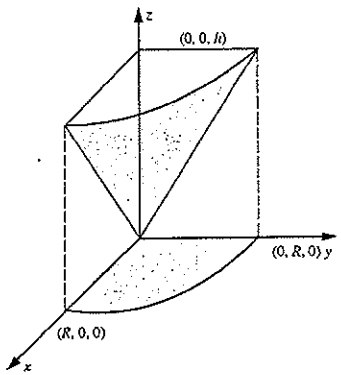
Centro de massa: $\left(0, 0, \frac{3}{4}h\right)$.

b) $I = \iiint_B (x^2 + y^2)\, dm = k\int_0^{2\pi}\int_0^R\int_{\frac{h\rho}{R}}^{h} \rho^2 \rho dz\, d\rho\, d\theta$

$$= \int_0^{2\pi}\left[\int_0^R \left(h\rho^3 - \frac{h\rho^4}{R}\right) d\rho\right] d\theta\, k = \frac{hR^4\pi k}{10}.$$

Então, $I = \frac{3}{10}R^2 M$, onde $M = \frac{\pi R^2 hk}{3}$ é a massa do cone.

10. Supondo que o eixo seja o eixo y,



$$I_y = \iiint_B r^2\, dm = \iiint_B (x^2 + z^2)\, k\, dx\, dy\, dz = k\iiint_{B_{\rho\theta z}} (\rho^2\cos^2\theta + z^2)\,\rho\, dz\, d\rho\, d\theta$$

$$= k\int_0^{2\pi}\int_0^R \left[\int_{\frac{h\rho}{R}}^{h} (\rho^3\cos^2\theta + \rho z^2)\, dz\right] d\rho\, d\theta = \frac{3M}{5}\left(h^2 + \frac{1}{4}R^2\right),$$

onde M é a massa do cone.

Então, $I_y = \frac{3M}{5}\left(h^2 + \frac{1}{4}R^2\right)$.

11. Vamos considerar a esfera com centro na origem e trabalhar em coordenadas cilíndricas.

$$\begin{cases} x = \rho\cos\theta \\ y = \rho\,\text{sen}\,\theta \\ z = z \end{cases} \qquad dx\, dy\, dz = \rho\, d\rho\, d\theta\, dz$$

$dm = \underbrace{\delta(x, y, z)}_{k\sqrt{x^2+y^2+z^2}}\, dx\, dy\, dz$, ou seja, $dm = k\sqrt{R^2 - \rho^2}\, dx\, dy\, dz$.

Temos

$$M = \iiint dm$$

$$= k\int_0^{2\pi}\int_0^R\int_0^{\sqrt{R^2-\rho^2}} \sqrt{R^2 - \rho^2}\, dz\,\rho\, d\rho\, d\theta$$

$$= k\int_0^{2\pi}\int_0^R (R^2 - \rho^2)\,\rho\, d\rho\, d\theta = \frac{k\pi R^4}{2}.$$

$$\iiint_B z\, dm = k\int_0^{2\pi}\int_0^R\int_0^{\sqrt{R^2-\rho^2}} \sqrt{R^2 - \rho^2}\, z\, dz\,\rho\, d\rho\, d\theta$$

$$= \frac{k}{2}\int_0^{2\pi}\int_0^R (R^2 - \rho^2)^{3/2}\,\rho\, d\rho\, d\theta = \frac{k\pi R^5}{5}.$$

$z_c = \frac{k\pi R^5}{5} \div \frac{k\pi R^4}{2} = \frac{2}{5}R$. Pela simetria do corpo $x_c = y_c = 0$.

Centro de massa $\left(0, 0, \frac{2}{5}R\right)$.

(Em coordenadas esféricas: $\iiint z\, dm = \int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^R k\rho^4\cos\varphi\,\text{sen}\,\varphi\, d\varphi\, d\theta =$

$$= k\left[\int_0^R \rho^4\, d\rho\right]\cdot\left[\int_0^{\frac{\pi}{2}} \cos\varphi \operatorname{sen}\varphi\, d\varphi\right]\left[\int_0^{2\pi} d\theta\right]$$

$$= k\left[\frac{\rho^5}{5}\right]_0^R\left[\frac{\operatorname{sen}^2\varphi}{2}\right]_0^{\frac{\pi}{2}}[\theta]_0^{2\pi} = \frac{k\pi R^5}{5}\Bigg).$$

12. Seja I_y o momento de inércia do cone circular reto de altura h e raio da base R em relação ao eixo y. (Exercício 10)

Seja r a reta que passa pelo centro de massa e é paralela ao eixo y.

$\left(\text{Centro de massa: } \left(0, 0, \frac{3}{4}h\right)\right)$

$$I_y = I_r + Md^2, \quad \text{onde} \quad d = \frac{3}{4}h$$

$$I_r = I_y - M\left(\frac{3}{4}h\right)^2$$

$$I_r = \frac{3M}{5}\left(h^2 + \frac{1}{4}R^2\right) - \frac{9}{16}Mh^2 \Rightarrow I_r = \frac{3M}{80}(h^2 + 4R^2).$$

Seja s o diâmetro da base do cone paralelo ao eixo dos y.
Pelo teorema de Steiner,

$$I_s = I_r + M\cdot\left(\frac{1}{4}h\right)^2 = \frac{3}{80}M(h^2 + 4R^2) + \frac{1}{16}h^2M = \frac{M}{20}(2h^2 + 3R^2).$$

CAPÍTULO 6

Exercícios 6.1

1. *a*) Sejam $\vec{F}(x, y, z) = x\vec{i} + y\vec{j} + z\vec{k}$ e $\gamma(t) = (\cos t, \operatorname{sen} t, t)$, $0 \leq t \leq 2\pi$.

Temos

$$\int_\gamma \vec{F}\, d\vec{r} = \int_0^{2\pi} \vec{F}(\gamma(t))\cdot\gamma'(t)\, dt = \int_0^{2\pi} (\cos t\,\vec{i} + \operatorname{sen} t\,\vec{j} + t\,\vec{k})\cdot(-\operatorname{sen} t, \cos t, 1)\, dt$$

$$= \int_0^{2\pi} (-\operatorname{sen} t\cos t + \operatorname{sen} t\cos t + t)\, dt = \int_0^{2\pi} t\, dt = \left[\frac{t^2}{2}\right]_0^{2\pi} = 2\pi^2.$$

b) Sejam $\vec{F}(x, y, z) = (x + y + z)\,\vec{k}$ e $\gamma(t) = (t, t, 1 - t^2)$, $0 \leq t \leq 1$.

Temos

$$\int_\gamma \vec{F}\, d\vec{r} = \int_0^1 F(g(t))\cdot\gamma'(t)\, dt = \int_0^1 (2t + 1 - t^2)\,\vec{k}\cdot(1, 1, -2t)\, dt$$

$$= \int_0^1 (-4t^2 - 2t + 2t^3)\, dt = \left[-\frac{4t^3}{3} - t^2 + \frac{t^4}{2}\right]_0^1 = -\frac{11}{6}.$$

d) Sejam $\vec{F}(x, y) = x^2\vec{i} + (x - y)\vec{j}$ e $\gamma(t) = (t, \operatorname{sen} t)$, $0 \leq t \leq \pi$.

Temos

$$\int_\gamma \vec{F}\, d\vec{r} = \int_0^\pi (\gamma(t))\cdot\gamma'(t)\, dt = \int_0^\pi (t^2\vec{i} + (t - \operatorname{sen} t)\vec{j})\cdot(1, \cos t)\, dt$$

$$= \int_0^\pi (t^2 + t\cos t - \operatorname{sen} t\cos t)\, dt = \int_0^\pi t^2\, dt + \int_0^\pi \underset{f}{t}\,\underset{g'}{\cos t}\, dt$$

$$- \int_0^\pi \operatorname{sen} t\cos t\, dt = \left[\frac{t^3}{3} + t\operatorname{sen} t + \cos t - \frac{\operatorname{sen}^2 t}{2}\right]_0^\pi$$

$$= \frac{\pi^3}{3} - 2.$$

2. Seja $\gamma(t) = (x(t), y(t))$, $a \leq t \leq b$, uma curva de classe C^1 com imagem contida na circunferência de centro na origem e raio r. Segue que $\gamma'(t)$ é tangente, no ponto $\gamma(t)$, à

curva $\gamma(t)$ e, portanto, ortogonal ao vetor $x(t)\vec{i} + y(t)\vec{j}$. Como $F(\vec{\gamma}(t))$ é paralelo a $x(t)\vec{i} + y(t)\vec{j}$, resulta que $F(\vec{\gamma}(t))$ é, também, ortogonal a $\gamma'(t)$ e daí $F(\gamma(t)) \cdot \gamma'(t) = \vec{0}$.

$$\int_\gamma \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\gamma(t)) \cdot \gamma'(t)\, dt = 0.$$

4. Seja $\vec{F}(x, y, z) = -y\vec{i} + x\vec{j} + z\vec{k}$.

a) $\gamma(t) = (\cos t, \operatorname{sen} t, t)$, $a = 0$ e $b = 2\pi$ ($\gamma(t)$ é a posição da partícula no instante t).
$F(\gamma(t)) = -\operatorname{sen} t\,\vec{i} + \cos t\,\vec{j} + t\,\vec{k}$ (força que age sobre a partícula no instante t).

$$\int_\gamma \vec{F}\, d\vec{r} = \int_a^b F(\gamma(t)) \cdot \gamma'(t)\, dt = \int_0^{2\pi} (-\operatorname{sen} t\,\vec{i} + \cos t\,\vec{j} + t\,\vec{k}) \cdot (-\operatorname{sen} t, \cos t, 1)\, dt$$

$$= \int_0^{2\pi} (\operatorname{sen}^2 t + \cos^2 t + t)\, dt = \int_0^{2\pi} (1 + t)\, dt = \left[t + \frac{t^2}{2} \right]_0^{2\pi} = 2\pi(1 + \pi).$$

b) $\gamma(t) = (2t + 1, t - 1, t)$, $a = 1$ e $b = 2$.

$$\int_\gamma \vec{F}\, d\vec{r} = \int_1^2 ((1 - t)\,\vec{i} + (2t + 1)\,\vec{j} + t\,\vec{k}) \cdot (2, 1, 1)\, dt$$

$$= \int_1^2 (2 - 2t + 2t + 1 + t)\, dt = \int_1^2 (3 + t)\, dt = \left[3t + \frac{t^2}{2} \right]_1^2 = \frac{9}{2}.$$

c) $\gamma(t) = (\cos t, 0, \operatorname{sen} t)$, $a = 0$ e $b = 2\pi$.

$$\int_\gamma \vec{F}\, d\vec{r} = \int_0^{2\pi} (\cos t\,\vec{j} + \operatorname{sen} t\,\vec{k}) \cdot (-\operatorname{sen} t, 0, \cos t)\, dt$$

$$= \int_0^{2\pi} \operatorname{sen} t \cos t\, dt = \left[\frac{\operatorname{sen}^2 t}{2} \right]_0^{2\pi} = 0.$$

5. Sejam $\vec{E}(x, y) = \dfrac{1}{x^2 + y^2} \dfrac{x\vec{i} + y\vec{j}}{\sqrt{x^2 + y^2}}$ e $\vec{l}(t) = \gamma(t) = (t, 1), -1 \leq t \leq 1$.

$$\int_\gamma \vec{E}\, d\vec{l} = \int_{-1}^{1} \left[\left(\frac{1}{t^2 + 1} \cdot \frac{t\vec{i} + \vec{j}}{\sqrt{t^2 + 1}} \right) \cdot (1, 0) \right] dt = \int_{-1}^{1} \underbrace{\frac{t}{(t^2 + 1)^{\frac{3}{2}}}}_{\text{função ímpar}}\, dt = 0.$$

6. Sejam $\vec{E}(x, y) = \dfrac{1}{x^2 + y^2} \dfrac{x\vec{i} + y\vec{j}}{\sqrt{x^2 + y^2}}$ e $\vec{l}(t) = \gamma(t) = (t, 1 - t^4), -1 \leq t \leq 1$.

A curva $\gamma(t) = (t, 1 - t^4)$, $-1 \leq t \leq 1$, é simétrica em relação ao eixo y. O campo $\vec{E}(x, y)$ é radial, ou seja, paralelo ao vetor $x\vec{i} + y\vec{j}$, além disso, a intensidade de $\vec{E}$ é a mesma em pontos simétricos em relação ao eixo y. É razoável, então, esperar que o trabalho realizado por $\vec{E}$ de (1, 0) a (0, 1) seja oposto ao trabalho realizado de (0, 1) a (−1, 0). Devemos esperar, então, que a integral seja zero.

b) $$\int_\gamma \vec{E}\, d\vec{l} = \int_{-1}^{1} \left(\frac{1}{t^2 + (1 - t^4)^2} \cdot \frac{t\vec{i} + (1 - t^4)\vec{j}}{\sqrt{t^2 + (1 - t^4)^2}} \right) \cdot (1, -4t^3)\, dt$$

$$= \int_{-1}^{1} \frac{t + (1 - t^4)(-4t^3)}{(t^2 + (1 - t^4)^2)^{3/2}}\, dt = 0.$$

7. Sejam $\vec{E}(x, y) = \dfrac{1}{x^2 + y^2} \dfrac{x\vec{i} + y\vec{j}}{\sqrt{x^2 + y^2}}$ e $\gamma(t) = (2 \cos t, \operatorname{sen} t)$, $0 \leq t \leq \dfrac{\pi}{2}$.

$$\int_\gamma \vec{E}\, d\vec{l} = \int_0^{\frac{\pi}{2}} \left(\frac{2 \cos t\,\vec{i} + \operatorname{sen} t\,\vec{j}}{(4 \cos^2 t + \operatorname{sen}^2 t)^{3/2}} \right) \cdot (-2 \operatorname{sen} t, \cos t)\, dt$$

$$= \int_0^{\frac{\pi}{2}} \frac{-4 \operatorname{sen} t \cos t + \operatorname{sen} t \cos t}{(4 \cos^2 t + \operatorname{sen}^2 t)^{3/2}}\, dt = \int_0^{\frac{\pi}{2}} -\frac{3 \operatorname{sen} t \cos t}{(4 \cos^2 t + \operatorname{sen}^2 t)^{3/2}}\, dt.$$

Façamos $u = 4 \cos^2 t + \operatorname{sen}^2 t$
$du = -6 \operatorname{sen} t \cos t\, dt$
$t = 0;\ u = 4$
$t = \dfrac{\pi}{2};\ u = 1.$

Temos, então:

$$\int_4^1 \left(\frac{1}{2} \right) \frac{du}{u^{3/2}} = \frac{1}{2} \left[\frac{u^{-\frac{1}{2}}}{-\frac{1}{2}} \right]_4^1 = (-1) \left[1 - \frac{1}{2} \right] = -\frac{1}{2}.$$

Exercícios 6.2

3. Parametrização do segmento de extremidades (0, 0, 0) e (1, 2, 1) no sentido de (1, 2, 1) para (0, 0, 0):

$(x, y, z) = (1, 2, 1) + t[(0, 0, 0) - (1, 2, 1)]$, $0 \leq t \leq 1$, ou seja,

$(x, y, z) = (1 - t, 2 - 2t, 1 - t)$, $0 \leq t \leq 1$.

Temos

$x(t) = 1 - t \Rightarrow \dfrac{dx}{dt} = -1,$

$y(t) = 2 - 2t \Rightarrow \dfrac{dy}{dt} = -2$ e

$z(t) = 1 - t \Rightarrow \dfrac{dz}{dt} = -1.$

$$\int_\gamma x\,dx + y\,dy + z\,dz = \int_0^1 \left[(1-t)\underbrace{\frac{dx}{dt}}_{(-1)} + (2-2t)\underbrace{\frac{dy}{dt}}_{(-2)} + (1-t)\underbrace{\frac{dz}{dt}}_{(-1)} \right] dt$$

$$= \int_0^1 \left[(t-1) + (4t-4) + (t-1) \right] dt = \int_0^1 (6t-6)\,dt = \left[\frac{6t^2}{2} - 6t \right]_0^1 = -3.$$

4. A projeção no plano xy da interseção do parabolóide $z = x^2 + y^2$ com o plano $z = 2x + 2y - 1$ é a circunferência $(x-1)^2 + (y-1)^2 = 1$, pois

$x^2 + y^2 = 2x + 2y - 1$

$\Leftrightarrow x^2 - 2x + 1 + y^2 - 2y + 1 = 1 \Leftrightarrow (x-1)^2 + (y-1)^2 = 1.$

A parametrização que atende às condições é:

$x - 1 = \cos t \Rightarrow x(t) = 1 + \cos t$ $\qquad (0 \leqslant t \leqslant 2\pi)$

$y - 1 = \operatorname{sen} t \Rightarrow y(t) = 1 + \operatorname{sen} t$

$z = 2\cos t + 2\operatorname{sen} t + 3 \Rightarrow z(t) = 2\cos t + 2\operatorname{sen} t + 3.$

Temos

$$\int_\gamma x\,dx + dy + 2\,dz = \int_0^{2\pi} \left[x(t)\frac{dx}{dt} + \frac{dy}{dt} + 2\frac{dz}{dt} \right] dt$$

$$= \int_0^{2\pi} \left[(1+\cos t)(-\operatorname{sen} t) + \cos t + 2(-2\operatorname{sen} t + 2\cos t) \right] dt$$

$$= \int_0^{2\pi} (-5\operatorname{sen} t + 5\cos t - \operatorname{sen} t \cos t)\,dt = \left[5\cos t + 5\operatorname{sen} t - \frac{\operatorname{sen}^2 t}{2} \right]_0^{2\pi} = 0.$$

5. A projeção no plano xz da interseção da esfera $x^2 + y^2 + z^2 = 2$ com o plano $y = x$ ($x \geqslant 0$, $y \geqslant 0$ e $z \geqslant 0$) é $2x^2 + z^2 = 2$ (elipse).

Em coordenadas esféricas

$$\begin{cases} x = \rho\cos\theta\operatorname{sen}\varphi \\ y = \rho\operatorname{sen}\theta\operatorname{sen}\varphi \\ z = \rho\cos\varphi \end{cases}$$

$y = x \Rightarrow \operatorname{sen}\theta = \cos\theta \Rightarrow \theta = \dfrac{\pi}{4}$

$2x^2 + z^2 = 2 \Rightarrow 2\rho^2 \operatorname{sen}^2\varphi\cos^2\theta + \rho^2\cos^2\varphi = 2$

$\Rightarrow 2\rho^2 \operatorname{sen}^2\varphi \left(\dfrac{\sqrt{2}}{2}\right)^2 + \rho^2\cos^2\varphi = 2 \Rightarrow \rho^2(\operatorname{sen}^2\varphi + \cos^2\varphi) = 2$

$\Rightarrow \rho = \sqrt{2}$

Logo, para se ter uma parametrização nas condições exigidas basta tomar as coordenadas esféricas com $\theta = \dfrac{\pi}{4}$ e $\rho = \sqrt{2}$.

O sentido do percurso é do ponto $(0, 0, \sqrt{2})$ para $(1, 1, 0)$.

Façamos $\operatorname{sen}\varphi = t$, $0 \leqslant \varphi \leqslant \dfrac{\pi}{2}$, $0 \leqslant t \leqslant 1$

Temos

$$\begin{cases} x(t) = t \\ y(t) = t \\ z(t) = \sqrt{2}\sqrt{1-t^2} \end{cases}$$

daí,

$\dfrac{dx}{dt} = 1,$ $\qquad \dfrac{dy}{dt} = 1$ $\qquad$ e $\qquad \dfrac{dz}{dt} = -\dfrac{\sqrt{2}\,t}{\sqrt{1-t^2}}.$

Portanto,

$$\int_\gamma dx + xy\,dy + z\,dz = \int_0^1 \left(\frac{dx}{dt} + x(t)\,y(t)\frac{dy}{dt} + z(t)\frac{dz}{dt} \right) dt$$

$$= \int_0^1 \left(1 + t^2 + \sqrt{2}\sqrt{1-t^2}\,\frac{-\sqrt{2}\,t}{\sqrt{1-t^2}} \right) dt = \int_0^1 (1 + t^2 - 2t)\,dt$$

$$= \left[t + \frac{t^3}{3} - t^2 \right]_0^1 = \frac{1}{3}.$$

6.

$\gamma: \begin{cases} x = 2\cos t \\ y = 2\operatorname{sen} t \end{cases}$ $\qquad 0 \leqslant t \leqslant \dfrac{\pi}{2}.$

$\dfrac{dx}{dt} = -2\operatorname{sen} t$ e $\dfrac{dy}{dt} = 2\cos t$. Então

$$\int_\gamma 2dx - dy = \int_0^{\frac{\pi}{2}} (-4\operatorname{sen} t - 2\cos t)\,dt = -6.$$

7. Parametrização da elipse $4x^2 + y^2 = 9$:

$$\begin{cases} \dfrac{2}{3}x = \cos t \;\Rightarrow\; x(t) = \dfrac{3}{2}\cos t \quad 0 \leq t \leq 2\pi \\ \dfrac{y}{3} = \operatorname{sen} t \;\Rightarrow\; y(t) = 3\operatorname{sen} t. \end{cases}$$

Temos

$$\int_\gamma -\frac{y}{4x^2+y^2}\,dx + \frac{x}{4x^2+y^2}\,dy = \int_0^{2\pi}\left[\left(-\frac{3\operatorname{sen} t}{9}\right)\left(-\frac{3}{2}\operatorname{sen} t\right) + \left(\frac{3\cos t}{18}\right)(3\cos t)\right]dt$$

$$= \int_0^{2\pi}\left(\frac{\operatorname{sen}^2 t}{2} + \frac{\cos^2 t}{2}\right)dt = \frac{1}{2}[t]_0^{2\pi} = \pi.$$

9. Uma parametrização bem natural que atende às condições dadas é

$$\begin{cases} x(t) = t \\ y(t) = t \\ z(t) = 2t^2 \end{cases} \qquad -1 \leq t \leq 1.$$

$$\frac{dx}{dt} = \frac{dy}{dt} = 1 \quad \text{e} \quad \frac{dz}{dt} = 4t.$$

$$\int_\gamma dx + y\,dy + dz = \int_{-1}^{1}\left(\frac{dx}{dt} + y(t)\frac{dy}{dt} + \frac{dz}{dt}\right)dt$$

$$= \int_{-1}^{1}(1 + t + 4t)\,dt = \int_{-1}^{1}(5t+1)\,dt = \left[\frac{5t^2}{2} + t\right]_{-1}^{1} = 2.$$

11. Uma parametrização bem natural que atende às condições dadas ($x^2 + 4y^2 = 1$, $x^2 + z^2 = 1$, $y \geq 0$ e $z \geq 0$) é:

$$\begin{cases} x(t) = \cos t \\ y(t) = \dfrac{\operatorname{sen} t}{2} \\ z(t) = \operatorname{sen} t \end{cases} \quad 0 \leq t \leq \pi.$$

$$\frac{dx}{dt} = -\operatorname{sen} t, \quad \frac{dy}{dt} = \frac{1}{2}\cos t \quad \text{e} \quad \frac{dz}{dt} = \cos t.$$

$$\int_\gamma 2y\,dx + z\,dy + x\,dz = \int_0^{\pi}\left(2y(t)\frac{dx}{dt} + z(t)\frac{dy}{dt} + x(t)\frac{dz}{dt}\right)dt$$

$$= \int_0^{\pi}\left[(\operatorname{sen} t)\cdot(-\operatorname{sen} t) + (\operatorname{sen} t)\left(\frac{1}{2}\cos t\right) + (\cos t)(\cos t)\right]dt$$

$$= \int_0^{\pi}\left(-\operatorname{sen}^2 t + \frac{1}{2}\operatorname{sen} t\cos t + \cos^2 t\right)dt$$

$$= \int_0^{\pi}\left(\cos 2t + \frac{\operatorname{sen} 2t}{4}\right)dt = \left[\frac{1}{2}\operatorname{sen} 2t - \frac{1}{8}\cos 2t\right]_0^{\pi} = 0.$$

Exercícios 6.3

1. *a)* $\gamma_2(u) = \gamma_1(g(u))$, onde $g(u) = \dfrac{u}{2}$, $0 \leq u \leq 2$. Como $g'(u) > 0$, $0 \leq u \leq 2$, e a imagem de g é o intervalo $[0, 1]$, segue que γ_2 é obtida de γ_1 por uma mudança de parâmetro que conserva a orientação, logo, as integrais sobre γ_1 e sobre γ_2 são iguais.

d) $\gamma_2(u) = \gamma_1(g(u))$, onde $g(u) = 1 - u$, $0 \leq u \leq 2$. Como $g'(u) < 0$, $0 \leq u \leq 2$, e a imagem de g é o intervalo $[-1, 1]$, segue que γ_2 é otida de γ_1 por uma mudança de parâmetro que reverte a orientação, logo, as integrais sobre γ_1 e sobre γ_2 têm valores opostos.

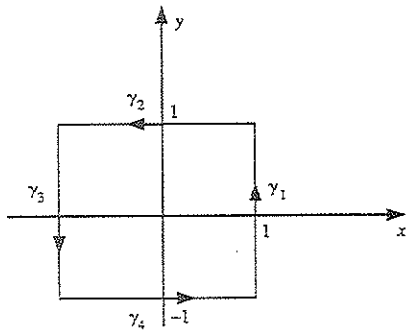
2. É falsa. Considere as curvas $\gamma_1(t) = (\cos t, \operatorname{sen} t)$, $0 \leq t \leq 2\pi$, e $\gamma_2(t) = (\cos t, \operatorname{sen} t)$, $0 \leq t \leq 4\pi$, e seja $\vec{F}(x, y) = -y\,\vec{i} + x\,\vec{j}$. As imagens de γ_1 e de γ_2 são iguais, pois ambas têm como imagem a circunferência de centro na origem e raio 1. Porém, as integrais são diferentes:

$$\int_{\gamma_1}\vec{F}\,d\vec{r} = \int_0^{2\pi}(\operatorname{sen}^2 t + \cos^2 t)\,dt = 2\pi \text{ e } \int_{\gamma_2}\vec{F}\cdot d\vec{r} = 4\pi.$$

Observe γ_2 não é obtida de γ_1 por qualquer mudança de parâmetro que conserva ou reverte a orientação. Supondo $\vec{F}$ um campo de forças (observe que este campo é tangente à circunferência), o trabalho realizado por $\vec{F}$ no deslocamento da partícula entre as posições $\gamma_1(0)$ e $\gamma_1(2\pi)$ é 2π e entre as posições $\gamma_2(0)$ e $\gamma_2(4\pi)$ é 4π.

Exercícios 6.4

2. Sejam $\vec{F}(x, y) = (x + y^2)\vec{j}$ e γ é a curva



Uma parametrização para γ é:

$$\gamma_1: \begin{cases} x = 1, & 0 \leq t \leq 1, \\ y = -1 + 2t \end{cases}$$

$$\gamma_2: \begin{cases} x = 1 - 2t, & 0 \leq t \leq 1, \\ y = 1 \end{cases}$$

$$\gamma_3: \begin{cases} x = -1, & 0 \leq t \leq 1, \text{ e} \\ y = 1 - 2t \end{cases}$$

$$\gamma_4: \begin{cases} x = -1 + 2t, & 0 \leq t \leq 1. \\ y = -1 \end{cases}$$

$$\int_\gamma \vec{F}\, d\vec{r} = \int_{\gamma_1} \vec{F}\, d\vec{r} + \int_{\gamma_2} \vec{F}\, d\vec{r} + \int_{\gamma_3} \vec{F}\, d\vec{r} + \int_{\gamma_4} \vec{F}\, d\vec{r}$$

Como $\int_{\gamma_2} \vec{F}\, d\vec{r} = 0$ e $\int_{\gamma_4} \vec{F}\, d\vec{r} = 0$ $(\gamma_2'(t) = (-2, 0)$ e $\gamma_4'(t) = (2, 0))$

temos

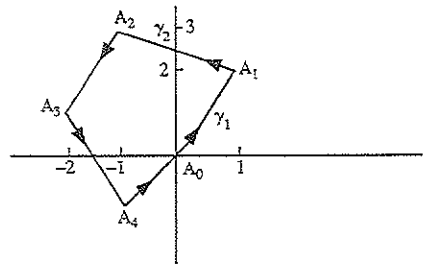
$$\int_\gamma \vec{F}\, d\vec{r} = \int_0^1 F(\gamma_1(t)) \cdot \gamma_1'(t)\, dt + \int_0^1 F(\gamma_3(t)) \cdot \gamma_3'(t)\, dt$$

$$= \int_0^1 \left[(1 + (-1 + 2t)^2)\vec{j} \cdot (0, 2)\right] dt + \int_0^1 \left[(-1 + (1 - 2t)^2)\vec{j} \cdot (0, -2)\right] dt$$

$$= \int_0^1 (4 - 8t + 8t^2)\, dt + \int_0^1 (8t - 8t^2)\, dt$$

$$= \left[4t - 4t^2 + \frac{8t^3}{3} + 4t^2 - \frac{8t^3}{3}\right]_0^1 = 4.$$

4. Seja γ a poligonal de vértices $A_0 = (0, 0)$, $A_1 = (1, 2)$; $A_2 = (-1, 3)$; $A_3 = (-2, 1)$ e $A_4 = (-1, -1)$ orientada de A_0 para A_4.



Uma parametrização para γ:

$$\gamma_1: \begin{cases} x = t, & 0 \leq t \leq 1, \\ y = 2t \end{cases}$$

$$\gamma_2: \begin{cases} x = 1 - 2t, & 0 \leq t \leq 1, \\ y = 2 + t \end{cases}$$

$$\gamma_3: \begin{cases} x = -1 - t, & 0 \leq t \leq 1, \\ y = 3 - 2t \end{cases}$$

$$\gamma_4: \begin{cases} x = -2 + t, & 0 \leq t \leq 1, \\ y = 1 - 2t \end{cases}$$

e

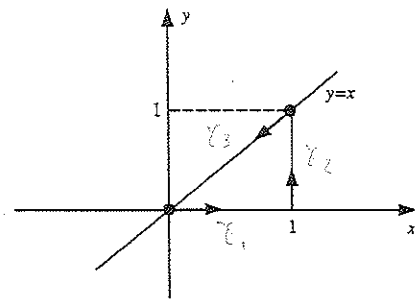
$$\gamma_5: \begin{cases} x = -1 + t, & 0 \leq t \leq 1. \\ y = -1 + t \end{cases}$$

Temos, então,

$$\int_\gamma dx + dy = \int_0^1 3\, dt + \int_0^1 -dt + \int_0^1 -3\, dt + \int_0^1 -3\, dt + \int_0^1 2\, dt$$

$$= -2 \int_0^1 dt = -2.$$

7. Sejam $P(x, y) = x^2 - y$ e $Q(x, y) = x^2 + y$.
B é o triângulo de vértices (0, 0), (1, 0) e (1, 1).



Uma parametrização de γ é:

γ_1: $\begin{cases} x = t, & 0 \leq t \leq 1, \\ y = 0 \end{cases}$

γ_2: $\begin{cases} x = 1, & 0 \leq t \leq 1, \\ y = t \end{cases}$

e

γ_3: $\begin{cases} x = 1 - t, & 0 \leq t \leq 1. \\ y = 1 - t \end{cases}$

Temos

$$\textcircled{1}\ \int_\gamma P\,dx + Q\,dy = \int_{\gamma_1} P\,dx + Q\,dy + \int_{\gamma_2} P\,dx + Q\,dy + \int_{\gamma_3} P\,dx + Q\,dy$$

$$= \int_0^1 t^2\,dt + \int_0^1 (1+t)\,dt + \int_0^1 (-2t^2 + 4t - 2)\,dt$$

$$= \left[\frac{t^3}{3} + t + \frac{t^2}{2} - \frac{2t^3}{3} + 2t^2 - 2t\right]_0^1 = \frac{1}{2} + \frac{2}{3} = \frac{7}{6}.$$

$$\textcircled{2}\ \iint_B \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \iint_B (2x+1)\,dx\,dy$$

$$= \int_0^1 \left[\int_0^x (2x+1)\,dy\right] dx = \int_0^1 [2xy + y]_0^x\,dx$$

$$= \int_0^1 (2x^2 + x)\,dx = \left[\frac{2x^3}{3} + \frac{x^2}{2}\right]_0^1 = \frac{2}{3} + \frac{1}{2} = \frac{7}{6}.$$

De ① e ② verificamos que

$$\int_\gamma P\,dx + Q\,dy = \iint_B \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy.$$

8. Supondo P e Q de classe C^1 num aberto Ω contendo o triângulo B de vértices (0, 0), (1, 0) e (1, 1) e γ é a fronteira de B parametrizada no Exercício 7.

Temos

$$\iint_B \frac{\partial Q}{\partial x}\,dx\,dy = \int_0^1 \int_y^1 \left(\frac{\partial Q}{\partial x}\,dx\right) dy$$

$$= \int_0^1 [Q(x, y)]_{x=y}^{x=1}\,dy = \int_0^1 [Q(1, y) - Q(y, y)]\,dy \quad \textcircled{1}$$

Por outro lado,

$$\int_\gamma Q\,dy = \int_{\gamma_1} Q(x, y)\,dy + \int_{\gamma_2} Q(x, y)\,dy = \int_{\gamma_3} Q(x, y)\,dy$$

$$= \int_0^1 Q(1, t)\,dt + \int_0^1 Q(1-t, 1-t)\,(-dt)$$

$$= \int_0^1 Q(1, t)\,dt - \int_0^1 Q(1-t, 1-t)\,dt = \int_0^1 Q(1, y)\,dy - \int_0^1 Q(y, y)\,dy$$

$$= \int_0^1 [Q(1, y) - Q(y, y)]\,dy \quad \textcircled{2}$$

Comparando ① e ② segue:

$$\iint_B \frac{\partial Q}{\partial x}\,dx\,dy = \int_\gamma Q\,dy. \quad \textcircled{3}$$

Analogamente, temos:

$$-\iint_B \frac{\partial P}{\partial y}\,dx\,dy = -\int_0^1 \int_0^x \left(\frac{\partial P}{\partial y}\,dy\right) dx$$

$$= -\int_0^1 [P(x, y)]_{y=0}^{y=x}\,dx = -\int_0^1 (P(x, x) - P(x, 0))\,dx$$

$$= \int_0^1 P(x, 0)\,dx - \int_0^1 P(x, x)\,dx. \quad \textcircled{4}$$

Por outro lado,

$$\int_\gamma P\,dx = \int_{\gamma_1} P\,dx + \int_{\gamma_2} P\,dx + \int_{\gamma_3} P\,dx = \int_0^1 P(t, 0)\,dt$$

$$+ \int_0^1 P(1-t, 1-t)\,(-dt) = \int_0^1 P(x, 0)\,dx - \int_0^1 P(x, x)\,dx. \quad \textcircled{5}$$

Comparando ④ e ⑤ segue:

$$-\iint_B \frac{\partial P}{\partial y}\,dx\,dy = \int_\gamma P\,dx. \quad \textcircled{6}$$

Somando ③ e ⑥, concluímos

$$\iint_B \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \int_\gamma P\,dx + Q\,dy.$$

10. Consideremos $f, g: [a, b] \to \mathbb{R}$ funções de classe C^1 tais que $\forall\, x \in [a, b], f(x) < g(x)$.

Seja $B = \{(x, y) \mid f(x) \leq y \leq g(x),\ a \leq x \leq b\}$

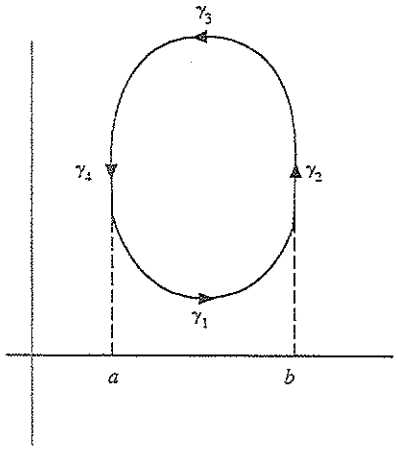
Seja P de classe C^1 num aberto que contém B.

Temos

$$\iint_B \frac{\partial P}{\partial y}\,dx\,dy = \int_a^b \left[\int_{f(x)}^{g(x)} \frac{\partial P}{\partial y}\,dy\right] dy$$

$$= \int_a^b \left[P(x,y)\right]_{y=f(x)}^{y=g(x)}\,dx = \int_a^b \left[P(x,g(x)) - P(x,f(x))\right] dx$$

$$= -\left[\int_a^b P(x,f(x))\,dx - \int_b^a P(x,g(x))\,dx\right].$$



Portanto,

$$-\iint_B \frac{\partial P}{\partial y}\,dx\,dy = \int_a^b P(x,f(x))\,dx - \int_a^b P(x,g(x))\,dx. \quad ①$$

$$\int_\gamma P\,dx = \int_a^b P(t,f(t))\,dt - \int_a^b P(t,g(t))\,dt$$

Por outro lado, considerando as curvas que formam a fronteira γ de B,

$$\gamma_1: \begin{cases} x = t \\ y = f(t) \end{cases} \qquad a \leq t \leq b,$$

$$\gamma_2: \begin{cases} x = b \\ y = t \end{cases} \qquad f(b) \leq t \leq g(b),$$

$$\gamma_3: \begin{cases} x = t \\ y = g(t) \end{cases} \qquad a \leq t \leq b,$$

e

$$\gamma_4: \begin{cases} x = a \\ y = f(a) + g(a) - t \end{cases} \qquad f(a) \leq t \leq g(a)$$

temos

$$\int_\gamma P\,dx = \int_{\gamma 1} P\,dx + \underbrace{\int_{\gamma 2} P\,dx}_{0} - \int_{\gamma 3} P\,dx + \underbrace{\int_{\gamma 4} P\,dx}_{0} \text{ e daí}$$

$$\int_\gamma P\,dx = \int_a^b P(t,f(t))\,dt - \int_a^b P(t,g(t))\,dt, \text{ ou seja,}$$

$$\int_\gamma P\,dx = \int_a^b P(x,f(x))\,dx - \int_a^b P(x,g(x))\,dx.$$

Comparando com ①, resulta o que queremos verificar.

11. Pelo Exercício 10, temos

$$\int_\gamma P\,dx = -\iint \frac{\partial P}{\partial y}\,dx\,dy.$$

Fazendo $P(x, y) = -y$

$$\int_\gamma -y\,dx = -\int \left[\int_B \frac{\partial(-y)}{\partial y}\,dy\right] dx = -\iint_B -dy\,dx = \iint_B dx\,dy$$

$$= \text{área de } B.$$

Portanto, área de $B = -\int_\gamma y\,dx$.

Exercícios 6.5

1. *c*) $\int_\gamma xy\,z\,ds = \int_0^{2\pi} t \operatorname{sen} t \cos t \underbrace{\|(-\operatorname{sen} t, \cos t, 1)\|}_{\sqrt{2}}\,dt$

$$= \sqrt{2}\int_0^{2\pi} \underbrace{t \operatorname{sen} t \cos t}_{\frac{1}{2}\operatorname{sen} 2t}\,dt = \frac{\sqrt{2}}{2}\int_0^{2\pi} \underset{f}{\underset{\downarrow}{t}} \cdot \underset{g''}{\underset{\downarrow}{\operatorname{sen} 2t}}\,dt$$

$$= \frac{\sqrt{2}}{2}\left[-\frac{t}{2}\cos 2t + \frac{1}{4}\operatorname{sen} 2t\right]_0^{2\pi} = -\pi\frac{\sqrt{2}}{2}.$$

3. Sejam $\gamma(t) = (\cos t, \operatorname{sen} t, t)$, $0 \leq t \leq \pi$, e
$\delta(x, y, z) = x^2 + y^2 + z^2$.

$$M = \int_\gamma \delta(x,y,z)\,ds = \int_0^\pi (\operatorname{sen}^2 t + \cos^2 t + t^2)\,\|(-\operatorname{sen} t, \cos t, 1)\|\,dt$$

$$= \sqrt{2}\int_0^\pi (1 + t^2)\,dt = \sqrt{2}\left[t + \frac{t^3}{3}\right]_0^\pi = \pi\sqrt{2}\left(1 + \frac{\pi^2}{3}\right).$$

5. Sejam $\gamma(t) = (t, 2t, 3t)$, $0 \leq t \leq 1$, e
$\delta(x, y, z) = x + y + z$.

$$I_z = \int_\gamma r^2\, dm, \text{ onde } r^2 = x^2 + y^2 \text{ e } dm = (x + y + z)\, ds.$$

$$I_z = \int_0^1 (5t^2)(6t) \underbrace{\|(1, 2, 3)\|}_{\sqrt{14}}\, dt = 30\sqrt{14} \int_0^1 t^3\, dt$$

$$= 30\sqrt{14}\left[\frac{t^4}{4}\right]_0^1 = \frac{30\sqrt{14}}{4} = \frac{15\sqrt{14}}{2}.$$

7. Seja $\gamma(t) = (\cos t, \operatorname{sen} t, t)$, $0 \leq t \leq \dfrac{\pi}{2}$.

$$I_x = \int_\gamma r^2\, dm = \int_\gamma (y^2 + z^2) \underbrace{\delta(x, y, z)}_{k \text{ (fio homogêneo)}} ds$$

$$= k\int_0^{\frac{\pi}{2}} (\operatorname{sen}^2 t + t^2)\|(-\operatorname{sen} t, \cos t, 1)\|\, dt$$

$$= \sqrt{2}\, k \int_0^{\frac{\pi}{2}} (\operatorname{sen}^2 t + t^2)\, dt$$

$$= \sqrt{2}\, k\left[-\frac{1}{2}\operatorname{sen} t \cos t + \frac{1}{2}t + \frac{t^3}{3}\right]_0^{\frac{\pi}{2}} = k\sqrt{2}\left[\frac{\pi}{4} + \frac{\pi^3}{24}\right]$$

$$= \frac{k\sqrt{2}\,\pi}{4}\left[1 + \frac{\pi^2}{6}\right].$$

8. *b*) Seja $\gamma(t) = (t, t^2, 0)$, $-1 \leq t \leq 1$.

$$M = \int_\gamma dm = \int_\gamma \underbrace{\delta(x, y, z)}_{k}\, ds = k\int_{-1}^1 \|(1, 2t, 0)\|\, dt$$

$$= k\int_{-1}^1 \sqrt{1 + 4t^2}\, dt = 2k\int_0^1 \sqrt{1 + 4t^2}\, dt.$$

Façamos a mudança de variável
$2t = tg\ \theta \Rightarrow 2\, dt = \sec^2\theta\, d\theta$
$t = 0;\ \theta = 0$
$t = 1;\ \theta = \operatorname{arctg} 2.$

Segue que

$$M = 2k\int_0^{\operatorname{arctg} 2} \sqrt{1 + \operatorname{tg}^2\theta}\, \frac{\sec^2\theta}{2}\, d\theta = k\int_0^{\operatorname{arctg} 2} \sec^3\theta\, d\theta$$

Fórmula de recorrência: $\displaystyle\int \sec^n\theta\, d\theta = \frac{1}{n-1}\sec^{n-2}\theta\operatorname{tg}\theta + \frac{n-2}{n-1}\int \sec^{n-2}\theta\, d\theta$

Portanto,

$$M = k\left[\frac{1}{2}\sec\theta\operatorname{tg}\theta + \frac{1}{2}\int_0^{\operatorname{arctg} 2}\sec\theta\, d\theta\right]_0^{\operatorname{arctg} 2}$$

$$= k\left[\frac{1}{2}\sec\theta\operatorname{tg}\theta + \frac{1}{2}\ln(\sec\theta + \operatorname{tg}\theta)\right]_0^{\operatorname{arctg} 2}.$$

Então,

$$M = \frac{k}{2}\left[2\sqrt{5} + \ln(2 + \sqrt{5})\right].$$

$$\int_\gamma x\, dm = \int_{-1}^1 kx\, ds = k\int_{-1}^1 \underbrace{t\sqrt{1 + 4t^2}}_{\text{função ímpar}}\, dt = 0.$$

$$\int_\gamma y\, dm = k\int_{-1}^1 t^2\sqrt{1 + 4t^2}\, dt = 2k\int_0^1 t^2\sqrt{1 + 4t^2}\, dt$$

$$= 2k\int_0^{\operatorname{arctg} 2} \frac{\operatorname{tg}^2\theta}{4} \cdot \underbrace{\sqrt{1 + \operatorname{tg}^2\theta}}_{\sec\theta} \cdot \frac{\sec^2\theta}{2}\, d\theta$$

$$= \frac{k}{4}\int_0^{\operatorname{arctg} 2} (\sec^2\theta - 1)\sec^3\theta\, d\theta = \frac{k}{4}\int_0^{\operatorname{arctg} 2} \sec^5\theta\, d\theta - \frac{k}{4}\int_0^{\operatorname{arctg} 2} \sec^3\theta\, d\theta$$

$$= \frac{k}{4}\left[\frac{1}{4}\sec^3\theta\operatorname{tg}\theta + \frac{3}{4}\int_0^{\operatorname{arctg} 2}\sec^3\theta\, d\theta\right]_0^{\operatorname{arctg} 2} - \frac{k}{4}\int_0^{\operatorname{arctg} 2}\sec^3\theta\, d\theta$$

$$= \left[\frac{k}{16}\sec^3\theta\operatorname{tg}\theta\right]_0^{\operatorname{arctg} 2} - \frac{k}{16}\left[\frac{1}{2}\sec\theta\operatorname{tg}\theta + \frac{1}{2}\ln(\sec\theta + \operatorname{tg}\theta)\right]_0^{\operatorname{arctg} 2}$$

$$= \frac{k}{16}\left(9\sqrt{5} - \frac{1}{2}\ln(\sqrt{5} + 2)\right).$$

$$y_c = \frac{\int_\gamma y\, dm}{\int_\gamma dm} = \frac{1}{8}\,\frac{\left[9\sqrt{5} - h(2 + \sqrt{5})/2\right]}{\left[2\sqrt{5} + \ln(2 + \sqrt{5})\right]}.$$

Centro de massa $(0, y_c, 0)$.

9. Sejam $\gamma(t) = (t, t, t)$, $0 \leq t \leq 1$ e $\delta(x, y, z) = xyz$.

$$\int_\gamma dm = \int_\gamma \delta(x, y, z)\, ds = \int_0^1 t^3 \,\|(1, 1, 1)\|\, dt = \sqrt{3}\int_0^1 t^3\, dt = \frac{\sqrt{3}}{4}.$$

$$\int_\gamma x\, dm = \int_0^1 t \cdot t^3 \cdot \sqrt{3}\, dt = \sqrt{3}\int_0^1 t^4\, dt = \frac{\sqrt{3}}{5}.$$

$$x_c = \frac{\int_\gamma x\, dm}{\int_\gamma dm} = \frac{\sqrt{3}}{5} \div \frac{\sqrt{3}}{4} = \frac{4}{5}.$$

Centro de massa $\left(\frac{4}{5}, \frac{4}{5}, \frac{4}{5}\right)$.

d) $\int \frac{x}{(x^2+y^2+z^2)^2}\, dx = \frac{1}{2}\int \frac{1}{u^2}\, du = \frac{1}{2}\int u^{-2} = -\frac{1}{2u} = -\frac{1}{2(x^2+y^2+z^2)}$

$u = x^2+y^2+z^2$

$du = 2x\, dx$

CAPÍTULO 7

Exercícios 7.1

1. *a*) Seja $\vec{F}(x, y, z, w) = (x, y, z, w)$.

Um campo vetorial $\vec{F}: \Omega \subset \mathbb{R}^4 \to \mathbb{R}^4$ é conservativo se existe um campo escalar diferenciável $\varphi: \Omega \subset \mathbb{R}^4 \to \mathbb{R}$, tal que $\nabla\varphi = \vec{F}$ em Ω.

Tomando-se $\varphi(x, y, z, w) = \frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} + \frac{w^2}{2}$, $\nabla\varphi = (x, y, z, w)$

tem-se $\nabla\varphi = \vec{F}$ em Ω. $(\Omega = \mathbb{R}^4)$.

Então, $\vec{F}$ é conservativo.

b) Seja $\vec{F}(x, y) = y\vec{i} + x\vec{j}$. $\nabla\varphi = (y, x)$

$\varphi(x, y) = xy$ é tal que $\nabla\varphi = \vec{F}$ em $\mathbb{R}^4$.

Então, $\vec{F}$ é conservativo.

c) Seja $\vec{F}(x, y, z) = \underbrace{(x - y)}_{P}\vec{i} + \underbrace{(x + y + z)}_{Q}\vec{j} + \underbrace{z^2}_{R}\vec{k}$.

Como $\vec{F}$ é de classe C^1 em $\mathbb{R}^2$, uma condição necessária para $\vec{F}$ ser conservativo é que

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x} \text{ e } \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}.$$

Como $\frac{\partial P}{\partial y} = -1$ e $\frac{\partial Q}{\partial x} = 1$, $\vec{F}$ não é conservativo.

d) Seja $\vec{F}(x, y, z) = \underbrace{\frac{x}{(x^2+y^2+z^2)^2}}_{P}\vec{i} + \underbrace{\frac{y}{(x^2+y^2+z^2)^2}}_{Q}\vec{j} + \underbrace{\frac{z}{(x^2+y^2+z^2)^2}}_{R}\vec{k}$.

Tomando-se $\varphi(x, y, z) = -\frac{1}{2(x^2+y^2+z^2)}$ tem-se $\nabla\varphi = \vec{F}$. Logo, $\vec{F}$ é conservativo.

e) Seja $\vec{F}(x, y, z) = x\vec{i} + y\vec{j} + z\vec{k}$.

Temos $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x} = \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y} = 0$.

$$\nabla\varphi = \frac{\partial\varphi}{\partial x}\vec{i} + \frac{\partial\varphi}{\partial y}\vec{j} + \frac{\partial\varphi}{\partial z}\vec{k}$$

Daí, segue

$$\frac{\partial\varphi}{\partial x} = x \cdot \frac{\partial\varphi}{\partial y} = y \quad \frac{\partial\varphi}{\partial z} = z.$$

Tomando-se $\varphi(x, y, z) = \frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2}$, tem-se $\Delta\varphi = \vec{F}$. Logo, $\vec{F}$ é conservativo.

2. Sejam $\vec{F}(x, y, z) = f(r)\frac{\vec{r}}{r}$, onde $f: \mathbb{R} \to \mathbb{R}$ é contínua,

$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ e $\|\vec{r}\| = r$.

Consideremos $\varphi(x, y, t) = g(\underbrace{\sqrt{x^2 + y^2 + z^2}}_{r})$, onde g é uma primitiva de f.

Temos:

$$\frac{\partial\varphi}{\partial x} = \frac{\partial g}{\partial r}\cdot\frac{\partial r}{\partial x} = \frac{dg}{dr}\cdot\frac{x}{\sqrt{x^2 + y^2 + z^2}},$$

$$\frac{\partial\varphi}{\partial y} = \frac{\partial g}{\partial r}\cdot\frac{\partial r}{\partial y} = \frac{dg}{dr}\cdot\frac{y}{\sqrt{x^2 + y^2 + z^2}} \text{ e}$$

$$\frac{\partial\varphi}{\partial z} = \frac{\partial g}{\partial r}\cdot\frac{\partial r}{\partial z} = \frac{dg}{dr}\cdot\frac{z}{\sqrt{x^2 + y^2 + z^2}}.$$

Então,

$$\nabla\varphi = \frac{x\vec{i}}{\sqrt{x^2 + y^2 + z^2}} f(r) + \frac{y\vec{j}}{\sqrt{x^2 + y^2 + z^2}} f(r) + \frac{z\vec{k}}{\sqrt{x^2 + y^2 + z^2}} f(r). \text{ Daí}$$

$$\nabla\varphi = \frac{x\vec{i} + y\vec{j} + z\vec{k}}{\underbrace{\sqrt{x^2 + y^2 + z^2}}_{\|r\| = r}} \cdot f(r) \text{ ou seja,}$$

$$\nabla\varphi = \frac{\vec{r}}{r} f(r) = \vec{F}. \text{ Logo, } \vec{F} \text{ é conservativo.}$$

Exercícios 7.2

1.

a) A forma diferencial $x\,dx + y\,dy + z\,dz$ é exata pois admite $\varphi(x, y, z)$

$= \frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2}$ como primitiva:

$$d\left(\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2}\right) = x\,dx + y\,dy + z\,dz.$$

c) A forma diferencial $yz\,dx + xz\,dy + xy\,dz$ é exata pois admite $\varphi(x, y, z) = xyz$ como primitiva.

e) A forma diferencial $(x + y)\,dx + (y - x)\,dy$ não é exata pois

$$\frac{\partial(x+y)}{\partial y} \neq \frac{\partial(y-x)}{\partial x} \left(\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}\right).$$

g) A forma diferencial $xy\,dx + y^2\,dy + xyz\,dz$ não é exata pois

$$\frac{\partial(xy)}{\partial y} \neq \frac{\partial(y^2)}{\partial x} \left(\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}\right).$$

h) Seja a forma diferencial $-\frac{y}{x^2 + y^2}\,dx + \frac{x}{x^2 + y^2}\,dy,\ y > 0$

onde $P(x, y) = -\frac{y}{x^2 + y^2}$ e $Q(x, y) = \frac{x}{x^2 + y^2}$ são funções de classe C^1 em

$\mathbb{R}^2 - \{(0, 0)\}$. Temos:

$$\left.\begin{aligned}\frac{\partial P}{\partial y} &= \frac{y^2 - x^2}{(x^2 + y^2)^2}\\ \frac{\partial Q}{\partial x} &= \frac{y^2 - x^2}{(x^2 + y^2)^2}\end{aligned}\right\}, \text{ ou seja, } \frac{\partial P}{\partial y}(x, y) = \frac{\partial Q}{\partial x}(x, y).$$

Como a condição necessária está verificada, a forma diferencial tem chance de ser exata.

Como $\theta(x, y) = -\text{arctg}\,\frac{x}{y}$, $y > 0$ é uma primitiva, segue que a forma diferencial é exata.

(Observamos que $\theta(x, y) = \frac{\pi}{2} - \text{arctg}\,\frac{x}{y}$, $y > 0$, é também uma primitiva, pois, funções com gradientes iguais, num conjunto conexo por caminhos, diferem, neste conjunto, por uma constante. Veja Seção 30.3 do Vol. 2.)

i) Seja $-\frac{y}{x^2+y^2}\,dx+\frac{x}{x^2+y^2}\,dy$, $(x, y)\in\Omega$ onde

$\Omega=\{(x, y)\in\mathbb{R}^2 \mid y>0\}\cup\{(x, y)\in\mathbb{R}^2 \mid x<0\}$.

$\theta_1(x, y)=\frac{\pi}{2}-\operatorname{arctg}\frac{x}{y}$, $y>0$, é uma primitiva de $-\frac{y}{x^2+y^2}\,dx+\frac{x}{x^2+y^2}\,dy$ no semiplano $y>0$. Por outro lado, $\theta_2(x, y)=\operatorname{arctg}\frac{y}{x}$, $x<0$, é uma primitiva no semiplano $x<0$. Então, no quadrante $x<0$ e $y>0$, $\theta_1(x, y)$ e $\theta_2(x, y)$ diferem por uma constante, ou seja,

$$\frac{\pi}{2}-\operatorname{arctg}\frac{x}{y}=\operatorname{arctg}\frac{y}{x}+k.$$

No ponto $(-1, 1)$, temos

$$\frac{\pi}{2}-\operatorname{arctg}\frac{-1}{1}=\operatorname{arctg}\frac{1}{-1}+k, \text{ ou seja, } \frac{\pi}{2}+\frac{\pi}{4}=-\frac{\pi}{4}+k$$

daí, $k=\pi$. Segue que

$$\theta(x, y)=\begin{cases}\frac{\pi}{2}-\operatorname{arctg}\frac{x}{y}, & y>0\\ \pi+\operatorname{arctg}\frac{y}{x}, & x<0\end{cases}$$

é uma primitiva da forma diferencial dada em Ω. Tal função pode ser dada de outra forma. Como $\theta_3(x, y)=k_1-\operatorname{arctg}\frac{x}{y}$ é uma primitiva da forma diferencial no semi-plano $y<0$, segue que no quadrante $x<0$ e $y<0$, $\theta_3(x, y)$ e $\theta_2(x, y)$ diferem por uma constante, logo, para algum valor de k_1 devemos ter

$$k_1-\operatorname{arctg}\frac{x}{y}=\pi+\operatorname{arctg}\frac{y}{x},\ x<0 \text{ e } y<0.$$

Fazendo $(x, y)=(-1, -1)$, obtemos $k_1-\frac{\pi}{4}=\pi+\frac{\pi}{4}$ e, portanto, $k_1=\frac{3\pi}{2}$. Segue que a primitiva $\theta(x, y)$ pode, também, ser dada assim

$$\theta(x, y)=\begin{cases}\frac{\pi}{2}-\operatorname{arctg}\frac{x}{y} & \text{se } y>0\\ \pi & \text{se } y=0 \text{ e } x<0\\ \frac{3\pi}{2}-\operatorname{arctg}\frac{x}{y} & \text{se } y<0.\end{cases}$$

2. Seja a forma diferencial $\underbrace{3x^{m+1}y^{n+1}}_{P(x, y)}\,dx+\underbrace{2x^{m+2}y^n}_{Q(x, y)}\,dy$.

Temos $\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}\Rightarrow 3(n+1)x^{m+1}y^n=2(m+2)x^{m+1}y^n\Rightarrow$

$\Rightarrow 3n+3=2m+4\Rightarrow\frac{3n+3}{m+2}=2.$

Com a condição $\frac{3n+3}{m+2}=2$, m e n naturais, $\varphi(x, y)=\frac{3}{m+2}x^{m+2}y^{n+1}$ é uma primitiva, em $\mathbb{R}^2$, da forma diferencial dada, pois, $\frac{\partial\varphi}{\partial x}=3x^{m+1}y^{n+1}$ e $\frac{\partial\varphi}{\partial y}=\frac{3n+3}{m+2}x^{m+2}y^n=2x^{m+2}y^n$. (**Observação.** Se permitirmos que m e n sejam reais quaisquer, com as condições $\frac{3n+3}{m+2}=2$ e $m\neq -2$, a função $\varphi(x, y)=\frac{3}{m+2}x^{m+2}y^{n+1}$, será uma primitiva no aberto $x>0$ e $y>0$. Se $m=-2$ e $n=-1$, $\varphi(x, y)=3\ln x+2\ln y$ será uma primitiva no aberto $x>0$ e $y>0$.)

4. Seja $\underbrace{(x^3+x+y)u}_{P}dx-\underbrace{xu}_{Q}dy$. Queremos determinar u que só dependa de x, $u=u(x)$, de modo que a forma diferencial seja exata. Temos

$\frac{\partial P}{dy}=u+(x^3+x+y)\frac{\partial u}{\partial y}=u$, pois, $\frac{\partial u}{\partial y}=0$ e $\frac{\partial Q}{\partial x}=-u-x\frac{du}{dx}$. Da condição necessária para a forma diferencial ser exata segue $u=-u-x\frac{du}{dx}$ que é uma equação diferencial de variáveis separáveis. Separando as variáveis, obtemos $\frac{du}{u}=-\frac{2}{x}dx$, $u\neq 0$ e $x\neq 0$. Integrando, resulta $\ln|u|=\ln\frac{k}{x^2}$, onde $k>0$ é a constante de integração. Deste modo, com $u=\frac{1}{x^2}$, $x\neq 0$, a condição necessária está verificada. Deixamos a seu cargo verificar que com este u a forma diferencial é exata.

5. Queremos determinar $u=u(y)$ que torne $(y^2+1)u\,dx+(x+y^2-1)u\,dy$ uma forma diferencial exata. Para que a condição necessária esteja verificada, devemos ter

$\frac{d}{\partial y}(y^2+1)u=\frac{d}{\partial x}(x+y^3-1)u$, ou seja, $2yu+(y^2+1)\frac{du}{dy}=u$ que é uma equação diferencial de variáveis separáveis. (*Lembre-se*: $\frac{\partial u}{\partial x}=0$, pois, u só depende de y.)

Separando as variáveis, obtemos $\frac{du}{u}=\frac{(1-2y)dx}{y^2+1}$, $u\neq 0$. Integrando, resulta $\ln|k\,u|=\operatorname{arctg} y-\ln(y^2+1)$, onde $k>0$ é a constante de integração. Deste modo, com $u=\frac{e^{\operatorname{arctg} y}}{y^2+1}$ a condição necessária está verificada. Fica a seu cargo verificar que com esta u a forma diferencial é exata.

Exercícios 7.3

1. *b*) $\int_\gamma y\,dx + x^2\,dy$ (a forma diferencial não é exata)

$\gamma(t) = (1, 1) + t[(2, 2) - (1, 1)] = (1 + t, 1 + t) \qquad 0 \leq t \leq 1$

Façamos $x(t) = 1 + t$ e $y(t) = 1 + t$.

Temos,

$$\int_\gamma y\,dx + x^2\,dy = \int_0^1 \left[(1+t) + (1+t)^2\right] dt = \int_0^1 (t^2 + 3t + 2)\,dt = \frac{23}{6}.$$

c) $\int_\gamma -\frac{y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy$, onde $\gamma:[0, 1] \to \mathbb{R}^2$ é uma curva C^1 por partes, com imagem contida no semiplano $y > 0$.

Pelo Exercício 1, *h* da Seção 7.2, $\theta(x, y) = \text{arctg}\,\frac{x}{y}$, $y > 0$ é uma primitiva e a forma é exata.

Então,

$$\int_\gamma -\frac{y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy = \left[\theta(x, y)\right]_{\gamma(0)}^{\gamma(1)} = \left[-\text{arctg}\,\frac{x}{y}\right]_{\gamma(0)=(1,1)}^{\gamma(1)=(-2,3)}$$

$$= -\text{arctg}\left(-\frac{2}{3}\right) + \text{arctg}\,1 = \frac{\pi}{4} + \text{arctg}\,\frac{2}{3}.$$

d) $\int_{(-1,0)}^{(1,0)} \frac{x}{x^2+y^2}\,dx + \frac{y}{x^2+y^2}\,dy$

$P(x, y) = \frac{x}{x^2+y^2}$ e $Q(x, y) = \frac{y}{x^2+y^2}$ são funções de classe C^1 em $\Omega = \mathbb{R}^2 - \{(0, 0)\}$.

De $\frac{\partial P}{\partial y} = -\frac{2xy}{(x^2+y^2)^2}$ e $\frac{\partial Q}{\partial x} = -\frac{2xy}{(x^2+y^2)^2}$ segue que é uma forma diferencial exata em Ω: $\varphi = \frac{1}{2}\ln(x^2+y^2) = \ln\sqrt{x^2+y^2}$ é uma primitiva em Ω.

Segue que:

$\int_{(-1,0)}^{(1,0)} \frac{x}{x^2+y^2}\,dx + \frac{y}{x^2+y^2}\,dy = \left[\ln\sqrt{x^2+y^2}\right]_{(-1,0)}^{(1,0)} = 0$ sendo a integral calculada sobre uma curva com imagem contida em Ω.

e) $\int_\gamma (\text{sen}\,xy + xy\cos xy)\,dx + x^2\cos xy\,dy$, onde

$\gamma(t) = (t^2 - 1, t^2 + 1)$, $-1 \leq t \leq 1$.

$$\left.\begin{aligned} \frac{\partial P}{\partial y} &= 2x\cos xy - x^2 y\,\text{sen}\,xy \\ \frac{\partial Q}{\partial x} &= 2x\cos xy - x^2 y\,\text{sen}\,xy \end{aligned}\right\} \Rightarrow \text{a forma diferencial é exata em } \mathbb{R}^2.$$

$\varphi(x, y) = x\,\text{sen}\,xy$ é uma primitiva e

$$\int_\gamma (\text{sen}\,xy + xy\cos xy)\,dx + x^2\cos xy\,dy = \left[x\,\text{sen}\,xy\right]_{\gamma(-1)}^{\gamma(1)} = 0.$$

f) $\int_\gamma -\frac{y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy$, onde $g:[0, 1] \to \mathbb{R}^2$ é uma curva C^1 por partes com imagem contida no conjunto

$\Omega = \{(x, y) \in \mathbb{R}^2 \mid y > 0\} \quad \{(x, y) \quad ^2 \quad x < 0\}$ tal que
$\gamma(0) = (1, 1)$ e $\gamma(1) = (-1, -1)$.

Pelo Exercício 1, *h* da Seção 7.2, vimos que a forma diferencial é exata em Ω. O campo vetorial é conservativo com função potencial assim definida:

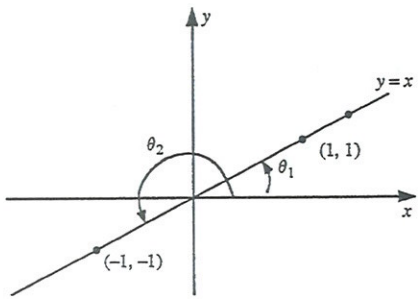
$$\theta(x, y) = \begin{cases} \frac{\pi}{2} - \text{arctg}\,\frac{x}{y} & \text{se } y > 0 \\ \pi & \text{se } y = 0 \text{ e } x < 0 \\ \frac{3\pi}{2} - \text{arctg}\,\frac{x}{y} & \text{se } y < 0. \end{cases}$$

Então,

$$\int_\gamma -\frac{y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy = \left[\theta(x, y)\right]_{(1,1)}^{(-1,-1)}$$

$$= \left[\theta(-1, -1) - \theta(1, 1)\right] = \left[\left(3\frac{\pi}{2} - \frac{\pi}{4}\right) - \left(\frac{\pi}{2} - \frac{\pi}{4}\right)\right] = \pi.$$

Observação. $\theta(x, y) = \theta_2 - \theta_1$, onde θ_1 e θ_2 são os ângulos assinalados na figura.



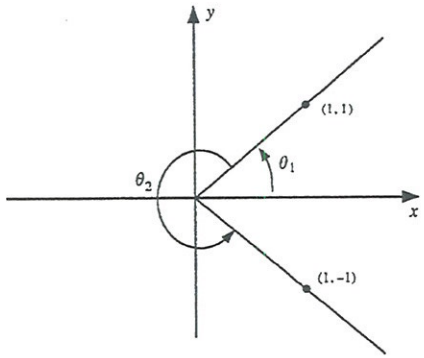
3. Analogamente ao Exercício 1, *f*.

$$\int_\gamma -\frac{y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy = [\theta(x,y)]_{\gamma(0)=(1,1)}^{\gamma(1)=(1,-1)}$$

$$= [\theta(1,-1) - \theta(1,1)] = \left[\frac{3\pi}{2} - \operatorname{arctg}(-1)\right]$$

$$-\left[\frac{\pi}{2} - \operatorname{arctg} 1\right] = \frac{3\pi}{2}$$

$$\left[\theta(x,y) = \frac{7\pi}{4} - \frac{\pi}{4} = \frac{3\pi}{2}\right].$$



Exercícios 7.6

1.

a) $h(x) = \int_1^{x^2} \operatorname{sen} t^2\,dt + \int_0^1 \frac{1}{1+xu^4}\,du.$

$h'(x) = \frac{d}{dx}\int_1^{x^2} \operatorname{sen} t^2\,dt + \frac{d}{dx}\int_0^1 \frac{1}{1+xu^4}\,du.$ Pelo Teorema Fundamental do

Cálculo, $\frac{d}{dx}\int_1^{x^2} \operatorname{sen} t^2\,dt = 2x \operatorname{sen} x^4$ e pelo que vimos na seção, temos

$\frac{d}{dx}\int_0^1 \frac{1}{1+xu^4}\,du = \int_0^1 \frac{\partial}{\partial x}\left(\frac{1}{1+xu^4}\right)du = \int_0^1 -\frac{u^4}{(1+xu^4)^2}\,du.$ Assim,

$$h'(x) = 2x \operatorname{sen} x^4 + \int_0^1 -\frac{u^4}{(1+xu^4)^2}\,du.$$

b) $h(x) = \int_0^1 \operatorname{sen}(x^2t^2)\,dt.$

$h'(x) = \frac{d}{dx}\int_0^1 \operatorname{sen}(x^2t^2)\,dt = \int_0^1 \frac{\partial}{\partial x}(\operatorname{sen}(x^2t^2))\,dt$

$= \int_0^1 2xt^2 \cos(x^2t^2)\,dt.$

c) $h(x) = \int_0^x \operatorname{sen}(x^2t^2)\,dt.$

Seja $\varphi(u, v) = \int_0^u \operatorname{sen}(v^2t^2)\,dt$. Temos

$\frac{\partial\varphi}{\partial u} = \operatorname{sen}(v^2u^2)$ e $\frac{\partial\varphi}{\partial v} = \int_0^u 2vt^2 \cos(v^2t^2)\,dt$. De

$h(x) = \varphi(u, v)$, $u = x$ e $v = x$, resulta, pela regra da cadeia,

$h'(x) = \frac{\partial\varphi}{\partial u}\frac{du}{dx} + \frac{\partial\varphi}{\partial v}\frac{dv}{dx}$ e, portanto,

$h'(x) = \operatorname{sen} x^4 + 2x\int_0^x t^2 \cos(x^2t^2)\,dt.$

d) $h(x) = \int_{x^2}^{\operatorname{sen} x} \frac{1}{1+x^4t^4}\,dt.$

Façamos $u = x^2$, $v = \operatorname{sen} x$ e $w = x^4$

Então, $h(x) = \varphi(u, v, w) = \int_u^v \underbrace{\frac{1}{1+wt^4}}_{f(w,t)}\,dt$

Pela regra da cadeia:

$$h'(x) = \frac{\partial\varphi}{\partial u}\frac{du}{dx} + \frac{\partial\varphi}{\partial v}\frac{dv}{dx} + \frac{\partial\varphi}{\partial w}\frac{dw}{dx}. \quad ①$$

Pelo teorema fundamental,

$$\frac{\partial\varphi}{\partial u} = \frac{\partial}{\partial u}\left[-\int_v^u f(w,t)\,dt\right] = -f(w,u) = -\frac{1}{1+wu^4}$$

$$= -\frac{1}{1+x^4\,(x^2)^4} = -\frac{1}{1+x^{12}};$$

$$\frac{\partial \varphi}{\partial v} = \frac{\partial}{\partial v}\int_u^v f(w,t)\,dt = f(w,v)$$

$$= \frac{1}{1+x^4(\operatorname{sen} x)^4};$$

$$\frac{\partial \varphi}{\partial w} = \frac{\partial}{\partial w}\int_u^v f(w,t)\,dt = \int_u^v \frac{\partial}{\partial w}\left(\frac{1}{1+wt^4}\right)dt$$

$$= \int_u^v -\frac{t^4}{(1+wt^4)^2}\,dt.$$

Temos $\frac{du}{dx} = 2x$, $\frac{dv}{dx} = \cos x$ e $\frac{dw}{dx} = 4x^3$.

Substituindo os valores encontrados em ①:

$$h'(x) = -\frac{2x}{1+x^{12}} + \frac{\cos x}{1+x^4(\operatorname{sen} x)^4} + 4x^3\int_{x^3}^{\operatorname{sen} x} -\frac{t^4}{(1+x^4t^4)^2}\,dt.$$

2. Supondo $\alpha(x)$ e $\beta(x)$ funções a valores reais diferenciáveis no intervalo aberto I; $f(x, y)$ de classe C^1 no aberto Ω de $\mathbb{R}^2$ e, para todo $x \in I$, o segmento de extremidades $(x, \alpha(x))$ e $(x, \beta(x))$ esteja contido em Ω.

Nestas condições, seja

$$h(x) = \int_{\alpha(x)}^{\beta(x)} f(x, y)\,dy, \quad x \in I.$$

Temos $h(x) = H(\alpha(x), \beta(x), z(x))$, onde $z = x$ e

$$H(\alpha, \beta, z) = \int_\alpha^\beta f(z, y)\,dy.$$

Pela regra da cadeia,

$$h'(x) = \frac{\partial H}{\partial \alpha}\frac{d\alpha}{dx} + \frac{\partial H}{\partial \beta}\frac{d\beta}{dx} + \frac{\partial H}{\partial y}\frac{dz}{dx}$$

ou seja,

$$h'(x) = \frac{\partial H}{\partial \alpha}\alpha'(x) + \frac{\partial H}{\partial \beta}\beta'(x) + \frac{\partial H}{\partial z}. \quad ①$$

Temos, em conseqüência do teorema fundamental:

$$\frac{\partial H}{\partial \alpha} = \frac{\partial}{\partial \alpha}\left[-\int_\beta^\alpha f(z, y)\,dy\right] = -f(z, \alpha(x))$$

$$\frac{\partial H}{\partial \beta} = \frac{\partial}{\partial \beta}\left[\int_\alpha^\beta f(z, y)\,dy\right] = -f(z, \beta(x))$$

$$\frac{\partial H}{\partial z} = \frac{\partial}{\partial z}\left[\int_\alpha^\beta f(z, y)\,dy\right] = \int_\alpha^\beta \frac{\partial}{\partial z} f(z, y)\,dy$$

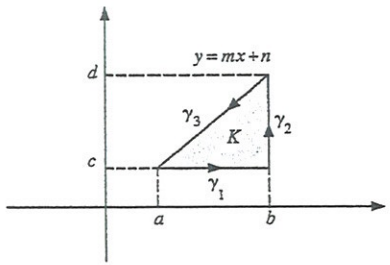
Substituindo em ①:

$$h'(x) = -f(x, \alpha(x))\,\alpha'(x) + f(x, \beta(x))\,\beta'(x) + \int_{\alpha(x)}^{\beta(x)} \frac{\partial f}{\partial x}(x, y)\,dy.$$

CAPÍTULO 8

Exercícios 8.1

4. Seja K o triângulo e γ a fronteira de k orientada no sentido anti-horário.



Vamos mostrar que

$$\oint_\gamma P dx + A dy = \iint_K \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy$$

Temos

$$\oint_\gamma P dx = \int_{\gamma_1} P(x, y)\, dx + \int_{\gamma_2} P(x, y)\, dx + \int_{\gamma_3} P(x, y)\, dx$$

onde (parametrização de γ)

$$\gamma_1: \begin{cases} x = t & a \leq t \leq b \quad (dx = dt) \\ y = c \end{cases}$$

$$\gamma_2: \begin{cases} x = b & c \leq t \leq d \quad (dx = 0) \\ y = t \end{cases}$$

$$\overline{\gamma}_3: \begin{cases} x = t & a \leq t \leq b \quad (dx = dt). \\ y = mt + n \end{cases}$$

(observe que $c = ma + n$ e $d = mb + n$)

Então,

$$\oint_\gamma P dx = \int_{\gamma_1} P(x, y)\, dx - \int_{\overline{\gamma}_3} P(x, y)\, dx = \int_a^b P(t; c)\, dt - \int_a^b P(t, mt + n)\, dt. \quad ①$$

Por outro lado,

$$\iint_K \frac{\partial P(x, y)}{\partial y} dx\, dy = \int_a^b \left[\int_c^{mx+n} \frac{\partial P(x, y)}{\partial y} dy \right] dx =$$

$$= \int_a^b \left[P(x, y) \right]_c^{mx+n} dx = \int_a^b \left[P(x, mx + n) - P(x, c) \right] dx \quad ②$$

De ① e ② resulta

$$\oint_\gamma P(x, y)\, dx = -\iint_K \frac{\partial P(x, y)}{\partial y} dx\, dy. \quad ③$$

De forma análoga,

$$\oint_\gamma Q(x, y)\, dy = \int_{\gamma_1} Q(x, y)\, dy + \int_{\gamma_2} Q(x, y)\, dy - \int_{\overline{\gamma}_3} Q(x, y)\, dy \text{ onde}$$

$$\gamma_1: \begin{cases} x = t \\ y = c \end{cases} \quad (dy = 0)$$

$$\gamma_2: \begin{cases} x = b & c \leq t \leq d \\ y = t \end{cases} \quad (dy = dt).$$

$$\overline{\gamma}_3: \begin{cases} x = \dfrac{t - n}{m} & c \leq t \leq d \\ y = t \end{cases} \quad (dy = dt).$$

(lembre de que $c = ma + n$ e $d = mb + n$)

Então,

$$\oint_\gamma Q(x, y)\, dy = \int_{\gamma_2} Q(x, y)\, dy - \int_{\overline{\gamma}_3} Q(x, y)\, dy =$$

$$= \int_c^d Q(b, t)\, dt - \int_c^d Q\left(\frac{t - n}{m}, t \right) dt. \quad ④$$

$$\iint_K \frac{\partial Q(x, y)}{\partial x} dx\, dy = \int_c^d \left[\int_{\frac{t-n}{m}}^b \frac{\partial Q(x, y)}{\partial x} dx \right] dy = \int_c^d \left[Q(x, y) \right]_{\frac{t-n}{m}}^b dy =$$

$$= \int_c^d \left[Q(b, y) - Q\left(\frac{t - n}{m}, y \right) \right] dy. \quad ⑤$$

De ④ e ⑤ resulta

$$\oint_\gamma Q(x, y)\, dy = \iint_K \frac{\partial Q(x, y)}{\partial x} dx\, dy. \quad ⑥$$

Somando ③ e ⑥, resulta

$$\oint P(x, y)\, dx + Q(x, y)\, dy = \iint_K \left(\frac{\partial Q(x, y)}{\partial x} - \frac{\partial P(x, y)}{\partial y} \right) dx\, dy.$$

Exercícios 8.2

1. Pelo Teorema de Green:

$$\oint P\,dx + Q\,dy = \iint_K \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy.$$

Façamos $P(x, y) = 0$ e $Q(x, y) = x$; $\dfrac{\partial Q}{\partial x} = 1$.

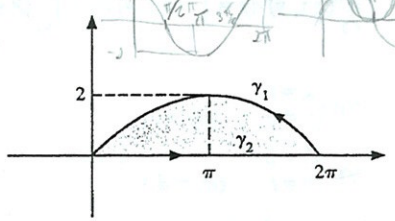
Então, temos:

$$\oint_\gamma x\,dy = \iint_K \frac{\partial Q}{\partial x}\,dx\,dy = \iint_K dx\,dy = \text{área de } K.$$

2.

$\bar{\gamma}_1$: $\begin{cases} x = t - \operatorname{sen} t & 0 \leq t \leq 2\pi \\ y = 1 - \cos t & (dy = \operatorname{sen} t\,dt) \end{cases}$

γ_2: $\begin{cases} x = t & 0 \leq t \leq 2\pi \\ y = 0 & (dy = 0) \end{cases}$



$$\text{Área} = -\int_{\bar{\gamma}_1} x\,dy + \int_{\gamma_2} x\,dy = -\int_0^{2\pi} (t - \operatorname{sen} t)\operatorname{sen} t\,dt + \int_0^{2\pi} t \cdot 0.$$

Daí,

$$\text{Área} = \int_0^{2\pi} (\operatorname{sen}^2 t - t \operatorname{sen} t)\,dt = 3\pi$$

3. Sejam $x = a\cos t$ e $y = b \operatorname{sen} t$, $0 \leq t \leq 2\pi$ onde $a > 0$ e $b > 0$ (elipse).

$$\text{Área} = \oint x\,dy = \int_0^{2\pi} (a\cos t)\cdot(b\cos t)\,dt =$$

$$= ab\int_0^{2\pi} \cos^2 t\,dt = ab\int_0^{2\pi} \left(\frac{1 + \cos 2t}{2}\right) dt =$$

$$= ab\left[t + \frac{1}{2}\operatorname{sen} 2t\right]_0^{2\pi} = \pi ab.$$

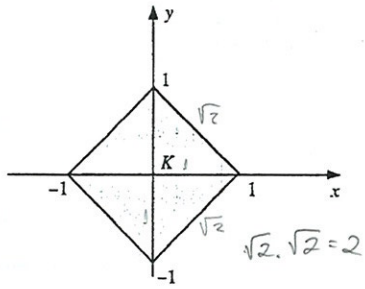
4. Seja $\vec{F}(x, y) = (2x + y)\,\vec{i} + (3x - y)\,\vec{j}$, onde

$P(x, y) = 2x + y$ $\left(\dfrac{\partial P}{\partial y} = 1\right)$, $Q(x, y) = 3x - y$ $\left(\dfrac{\partial Q}{\partial x} = 3\right)$.

$$\oint_\gamma \vec{F}\cdot d\gamma = \oint_\gamma P\,dx + Q\,dy = \iint_B \underbrace{\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)}_{2} dx\,dy =$$

$$= 2\iint_B dx\,dy = 2\cdot\underbrace{(\text{área de } B)}_{\alpha} = 2\alpha.$$

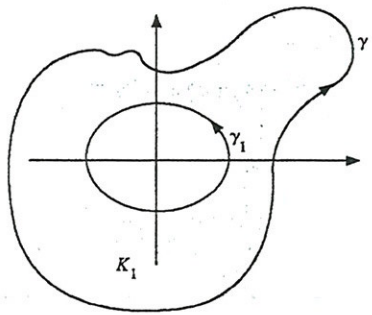
5. Seja $\vec{F}(x, y) = \underbrace{4x^3y^3}_{P(x,y)}\,\vec{i} + \underbrace{(3x^4y^2 + 5x)}_{Q(x,y)}\,\vec{j}$.



$$\frac{\partial P}{\partial y} = 12x^3y^2 \quad \text{e} \quad \frac{\partial Q}{\partial x} = 12x^3y^2 + 5.$$

$$\oint_\gamma \vec{F}\cdot d\gamma = \iint_K \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \iint_K 5\,dx\,dy = 5\cdot\underbrace{(\text{área de } K)}_{2} = 10$$

6. O teorema de Green não se aplica, pois, $P(x, y) = -\dfrac{y}{x^2 + y^2}$ e $Q(x, y) = \dfrac{x}{x^2 + y^2}$ não estão definidas em (0, 0) que pertence ao compacto K que admite γ como fronteira.



γ_1: $\begin{cases} x = \cos t \\ y = \operatorname{sen} t \end{cases}$ $0 \leq t \leq 2\pi$.

K_1 é a região limitada pelas curvas γ e γ_1. Pelo Exemplo 3,

$$\oint_\gamma P\,dx + Q\,dy - \oint_{\gamma_1} P\,dx + Q\,dy = \iint_{K_1}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\,dy.$$

De

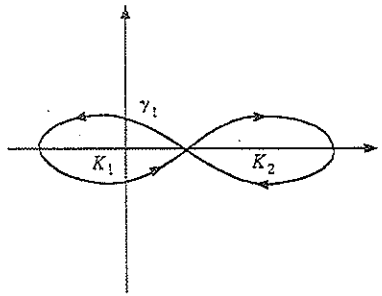
$$\oint_{\gamma_1} P\,dx + Q\,dy = \int_0^{2\pi} (\operatorname{sen}^2 t + \cos^2 t)\,dt = 2\pi$$

e $\iint_K \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\,dy = 0$ (verifique)

resulta

$$\oint_\gamma P\,dx + Q\,dy = 2\pi.$$

7.



Seja γ_1 a fronteira de K_1 orientada no sentido anti-horário. Pelo exercício anterior

$$\oint_{\gamma_1} -\frac{y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy = 2\pi.$$

Seja γ_2 a fronteira de K_2 orientada no sentido anti-horário. Nesta região, o teorema de Green se aplica. De

$$\iint_{K_2}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\,dy = 0$$

segue que

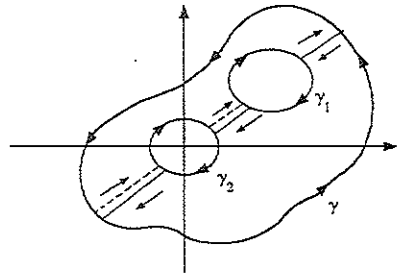
$$\oint_{\gamma_2} P\,dx + Q\,dy = 0.$$

Logo,

$$\oint_\gamma \underbrace{-\frac{y}{x^2+y^2}}_{P}\,dx + \underbrace{\frac{x}{x^2+y^2}}_{Q}\,dy = 2\pi.$$

8. P e Q são de classe C^1 em $\Omega = \mathbb{R}^2 - \{(0,0),(1,1)\}$. Seja B a região compreendida entre as curvas γ, γ_1 e γ_2. Portanto, P e Q são de classe C^1 num aberto contendo B. A fronteira de B consiste em γ, γ_1, γ_2. Raciocinando como no Exemplo 3, temos

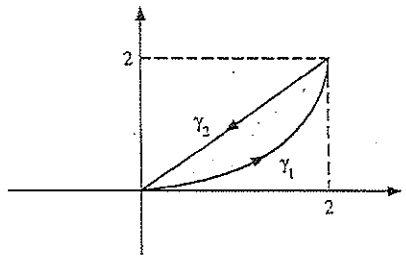
$$\oint_\gamma P\,dx + Q\,dy + \oint_{\gamma_1} P\,dx + Q\,dy + \oint_{\gamma_2} P\,dx + Q\,dy = \underbrace{\iint_B \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\,dy}_{0}$$



Portanto,

$$\oint_\gamma P\,dx + Q\,dy = -\oint_{\gamma_1} P\,dx + Q\,dy - \oint_{\gamma_2} P\,dx + Q\,dy = -3.$$

9. Inicialmente, observamos que, para $0 < t < 1$, $t^5 + t < t^3 + t$, o que significa que o ponto $(t^3 + t, t^5 + t)$, para $0 < t < 1$, permanece abaixo da reta $y = x$. Seja K a região limitada pelas curvas dadas.



$\gamma_1: \begin{cases} x = t^3 + t \\ y = t^5 + t \end{cases} \qquad 0 \leq t \leq 1$

$\bar{\gamma}_2: \begin{cases} x = t \\ y = t \end{cases} \qquad 0 \leq t \leq 2.$

Então, área de $K = \oint_\gamma x\,dy = \int_{\gamma_1} x\,dy - \int_{\gamma_2} x\,dy$, ou seja,

área de $K = \int_0^1 (t^3 + t)(5t^4 + 1)\,dt - \int_0^2 t\,dt = \frac{5}{24}$.

Exercícios 8.4

1. *a*) Sejam $\vec{F}(x, y) = x\vec{i} + y\vec{j}$, $\gamma(t) = (\cos t, \operatorname{sen} t)$, $0 \leq t \leq 2\pi$ e $\vec{n}$ a normal exterior.

$\oint \vec{F} \cdot \vec{n}\,ds = \iint_K \operatorname{div} \vec{F}\,dx\,dy$, onde K é o círculo $x^2 + y^2 \leq 1$.

$\operatorname{div} \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} = 1 + 1 = 2.$

Então,

$$\oint \vec{F} \cdot \vec{n}\,ds = \iint_K 2\,dx\,dy = 2\int_0^{2\pi}\int_0^1 \rho\,d\rho\,dt = 2\int_0^{2\pi}\left[\frac{\rho^2}{2}\right]_0^1 dt = \int_0^{2\pi} dt = 2\pi.$$

c) Sejam $\vec{F}(x, y) = x^2\vec{i}$, $\gamma(t) = (2\cos t, \operatorname{sen} t\}$, $0 \leq t \leq 2\pi$, e

$K = \left\{(x, y) \in \mathbb{R}^2 \mid \frac{x^2}{4} + y^2 \leq 1\right\}$.

Então,

$$\oint_\gamma \vec{F} \cdot \vec{n}\,ds = \iint_K \operatorname{div} F\,dx\,dy = \iint_K 2x\,dx\,dy =$$

$$= \int_0^{2\pi}\int_0^1 2(2\cos t)\,\rho\,d\rho\,dt = 4\int_0^{2\pi} \cos t \left[\frac{\rho^2}{2}\right]_0^1 dt = 2\int_0^{2\pi} \cos t\,dt = 0$$

Observe que pela simetria do campo e da região em relação ao eixo y, o fluxo através da fronteira localizada no semiplano $x < 0$ deve ser igual ao fluxo que sai pela fronteira localizada no semiplano $x > 0$. É razoável então esperar que o fluxo através da fronteira seja nulo.

d) $\vec{F}(x, y) = x^2\vec{i}$ e $y(t) = (2\cos t, \operatorname{sen} t)$, $0 \leq t \leq \pi$, e $\vec{n}$ a normal com componente $y \geq 0$. Observe que o fluxo através de γ é igual ao fluxo através da fronteira, orientada no sentido anti-horário e com normal apontando para fora, do conjunto limitado pela curva

$\frac{x^2}{4} + y^2 = 1$, $y \geq 0$, e pelo eixo x, pois, sendo $\vec{F}$ ortogonal a $-\vec{j}$, o fluxo através do segmento $y = 0$, $0 \leq x \leq 2$ é nulo. Então, pelo teorema da divergência,

$$\int_\gamma \vec{F}(x, y) \cdot \vec{n}\,ds = \iint_K \operatorname{div} \vec{F}\,dxdy$$

onde $K = \left\{(x, y) \in \mathbb{R}^2 \mid \frac{x^2}{4} + y^2 \leq 1 \text{ e } y \geq 0\right\}$.

Daí, $\int_\gamma \vec{F} \cdot \vec{n}\,ds = \int_{-2}^{2}\int_0^{\sqrt{1-\frac{x^2}{4}}} x\,dy\,dx = \int_{-2}^{2} x\sqrt{1-\frac{x^2}{4}}\,dx = 0$

pois o integrando é função ímpar.

e) Sejam $\vec{F}(x, y) = x\vec{i} + y\vec{j}$; $\gamma(t) = (t, t^2)$, $0 \leq t \leq 1$ e $\vec{n}$ a normal com componente $y < 0$.

Temos $\gamma'(t) = (1, 2t)$ e $\| \gamma'(t) \| = \sqrt{1+4t^2}$;

$\vec{n}(\gamma(t)) = \frac{1}{\| \gamma'(t) \|}(y'(t)\,\vec{i} - x'(t)\,\vec{j})$, ou seja,

$\vec{n}(\gamma(t)) = \frac{2t}{\sqrt{1+4t^2}}\vec{i} - \underbrace{\frac{1}{\sqrt{1+4t^2}}}_{\text{componente } y < 0}\vec{j}.$

$$\oint_\gamma \vec{F} \cdot \vec{n}\,ds = \int_0^1 \vec{F}(\gamma(t)) \cdot \vec{n}(\gamma(t)) \| \gamma'(t) \|\,dt =$$

$$= \int_0^1 (t\vec{i} + t^2\vec{j}) \cdot \left(\frac{2t}{\sqrt{1+4t^2}}\vec{i} - \frac{1}{\sqrt{1+4t^2}}\vec{j}\right) \| \gamma'(t) \|\,dt =$$

$$= \int_0^1 (2t^2 - t^2)\,dt = \frac{1}{3}.$$

2. Supondo γ definida em $[a, b]$ e sendo $\vec{F} \cdot \vec{n}$ constante, teremos

$$\int_\gamma \vec{F} \cdot \vec{n}\,ds = \vec{F} \cdot \vec{n} \int_a^b \underbrace{\| \gamma'(t) \|\,dt}_{ds},$$

ou seja, o fluxo de $\vec{F}$ através de γ é o produto de $\vec{F} \cdot \vec{n}$ pelo comprimento de γ.

3. Seja $\vec{F}(x, y) = \frac{x}{(x^2 + y^2)^5}\vec{i} + \frac{y}{(x^2 + y^2)^5}\vec{j}$.

Temos:

$$\gamma(t) = (\cos t, \operatorname{sen} t),\ 0 \leqslant t \leqslant \pi,$$
$$\gamma'(t) = (-\operatorname{sen} t, \cos t) \text{ e } \|\gamma'(t)\| = \|(-\operatorname{sen} t, \cos t)\| = 1.$$

$$\vec{n}(\gamma(t)) = \cos t\, \vec{i} + \operatorname{sen} t\, \vec{j}.$$

Calculemos o produto $\vec{F} \cdot \vec{n}$:

$$\vec{F}(\gamma(t)) \cdot \vec{n}(\gamma(t)) = (\cos t\, \vec{i} + \operatorname{sen} t\, \vec{j}) \cdot (\cos t\, \vec{i} + \operatorname{sen} t\, \vec{j}) = 1,$$

ou seja, $\vec{F} \cdot \vec{n} = 1$ (constante). Então,

$$\int_\gamma \vec{F} \cdot \vec{n}\, ds = \int_0^\pi \underbrace{\vec{F} \cdot \vec{n}}_{1}\ \underbrace{\|\gamma'(t)\|}_{1}\, dt = \pi.$$

6. Seja $\vec{F}(x, y) = \dfrac{x}{(x^2+y^2)^\alpha}\, \vec{i} + \dfrac{y}{(x^2+y^2)^\alpha}\, \vec{j}$.

F é solenoidal $\Leftrightarrow$ div $\vec{F} = 0$.

$$\operatorname{div} \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} = \frac{(x^2+y^2)^\alpha - 2\alpha x^2 (x^2+y^2)^{\alpha-1}}{(x^2+y^2)^{2\alpha}} +$$
$$+ \frac{(x^2+y^2)^\alpha - 2\alpha y^2 (x^2+y^2)^{\alpha-1}}{(x^2+y^2)^{2\alpha}} = \frac{2(1-\alpha)}{(x^2+y^2)^\alpha}$$

Então,

$$\operatorname{div} \vec{F} = 0 \iff 1 - \alpha = 0 \Leftrightarrow \alpha = 1.$$

Logo, $\vec{F}(x, y) = \dfrac{x}{x^2+y^2}\, \vec{i} + \dfrac{y}{x^2+y^2}\, \vec{j}$ é solenoidal.

Para desenho do campo, veja Exercício 1g da Seção 1.2.

7. *a*) As condições do teorema da divergência estão estabelecidas no enunciado. Temos

$$\oint_\gamma \frac{\partial g}{\partial \vec{n}}\, ds = \oint_\gamma \nabla g \cdot \vec{n}\, ds = \iint_K \operatorname{div}(\nabla g)\, dx\, dy =$$
$$= \iint_K \left(\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} \right) dx\, dy = \iint_K \nabla^2 g\, dx\, dy$$

pois $\dfrac{\partial g}{\partial \vec{n}}$ é a derivada direcional de g na direção de $\vec{n}$, ou seja, $\dfrac{\partial g}{\partial \vec{n}} = \nabla g \cdot \vec{n}$ e $\nabla^2 g$ é o laplaciano de g.

b) $\oint_\gamma f \dfrac{\partial g}{\partial \vec{n}}\, ds = \oint_\gamma f \cdot \nabla g \cdot \vec{n}\, ds = \iint_K \operatorname{div}(f \cdot \nabla g)\, dx\, dy$

Pelo Exercício 9c da Seção 1.4 deste volume,

$$\operatorname{div}(f \cdot \nabla g) = f \cdot \underbrace{\operatorname{div}(\nabla g)}_{\nabla^2 g} + \nabla f \cdot \nabla g.$$

Então,

$$\iint_K \operatorname{div}(f \cdot \nabla g)\, dx\, dy = \iint_K f \cdot \nabla^2 g\, dx\, dy + \iint_K (\nabla f \cdot \nabla g)\, dx\, dy$$

c) $\oint_\gamma f \cdot \dfrac{\partial f}{\partial \vec{n}}\, ds = \oint_\gamma f \cdot \nabla f \cdot \vec{n}\, ds = \iint_K \operatorname{div}(f \cdot \nabla f)\, dx\, dy =$

$= \iint_K (f \operatorname{div} \nabla f + \nabla f \cdot \nabla f)\, dx\, dy = \iint_K (f \cdot \nabla^2 f + \|\nabla f\|^2)\, dx\, dy.$

8. Pelo item *c* do exercício anterior,

$$\oint_\gamma v \frac{dv}{d\vec{n}}\, ds = \iint_K (v \nabla^2 v + \|\nabla v\|^2)\, dx\, dy.$$

Da hipótese $v(\gamma(t)) = 0$, $a \leqslant t \leqslant b$, segue que a integral do primeiro membro é nula e, tendo em vista a hipótese $\nabla^2 v = 0$ no interior de K, resulta

$\iint_K \|\nabla v\|^2\, dx\, dy = 0$. Da continuidade de ∇v, segue que $\nabla v(x, y) = 0$ no interior de K. Como o interior de K é aberto conexo por caminhos, existe uma constante c tal que $v(x, y) = c$ para todo (x, y) no interior de K. De $v(\gamma(t)) = 0$, $a \leqslant t \leqslant b$, resulta $c = 0$. Assim $v(x, y) = 0$ para todo (x, y) em K.

9. Utilize o Exercício 8, com $v = u_2 - u_1$.

$$\int_\gamma \vec{F} \cdot \vec{n}\, ds = \int_c^d \vec{F}(\gamma(g(u))) \cdot \vec{n}(\gamma(g(u)))\, \|\gamma'(g(u))\|\, \gamma'(u) du.$$

Lembrando que $\delta(u) = \gamma(g(u))$, $\delta'(u) = \gamma'(g(u))g'(u)$ e $g'(u) > 0$ resulta

$$\int_\gamma \vec{F} \cdot \vec{n}\, ds = \int_c^d \vec{F}(\delta(u)) \cdot \vec{n}(\delta(u)) \|\delta'(u)\|\, du = \int_\delta \vec{F} \cdot \vec{n}\, ds.$$

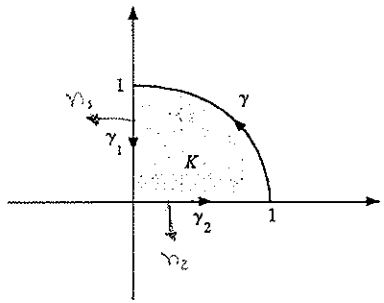
No caso $g'(u) < 0$ (mudança com reversão na orientação), $g(c) = b$ e $g(d) = a$, temos

$$\int_\gamma \vec{F}\cdot\vec{n}\,ds = -\int_d^c \vec{F}(\gamma(g(u)))\cdot\vec{n}(\gamma(g(u)))\|\gamma'(g(u))\|\,(-g'(u))\,du$$

e, portanto,

$$\int_\gamma \vec{F}\cdot\vec{n}\,ds = \int_c^d \vec{F}(\delta(u))\cdot\vec{n}(\delta(u))\|\delta'(u)\| = \int_\delta \vec{F}\cdot\vec{n}\,ds.$$

11.



$$\gamma: \begin{cases} x = \cos t \\ y = \operatorname{sen} t \end{cases} \quad 0 \leq t \leq \frac{\pi}{2},$$

$$\bar{\gamma}_1: \begin{cases} x = 0 \\ y = t \end{cases} \quad 0 \leq t \leq 1, \text{ e}$$

$$\gamma_2: \begin{cases} x = t \\ y = 0 \end{cases} \quad 0 \leq t \leq 1.$$

Sendo γ_3 a fronteira de K, orientada no sentido anti-horário, pelo teorema da divergência

$$\oint_{\gamma_3} \vec{F}\cdot\vec{n}\,ds = \iint_K \underbrace{\operatorname{div}\vec{F}}_{3}\,dxdy = \frac{3\pi}{4}$$

pois a área de K é $\frac{\pi}{4}$ e div $\vec{F} = 3$. Temos

$$\oint_{\gamma_3} \vec{F}\cdot\vec{n}\,ds = \int_\gamma \vec{F}\cdot\vec{n}\,ds + \int_{\gamma_1} \vec{F}\cdot\vec{n}\,ds + \int_{\gamma_2} \vec{F}\cdot\vec{n}\,ds.$$

Temos, ainda,

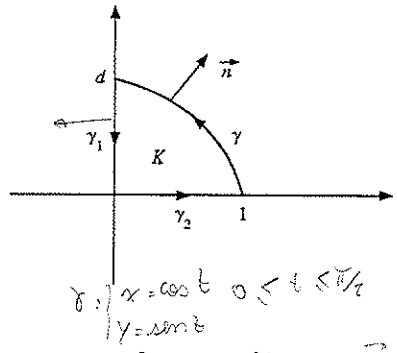
$$\int_{\gamma_1} \vec{F}\cdot\vec{n}\,ds = -\int_0^1 \vec{F}(0,t)\cdot(-\vec{i})\,dt = -\int_0^1 (3t\vec{j})\cdot(-\vec{i})\,dt = 0,$$

$$\int_{\gamma_2} \vec{F}\cdot\vec{n}\,ds = \int_0^1 \vec{F}(t,0)\cdot(-\vec{j})\,dt = 0.$$

Segue que

$$\int_\gamma \vec{F}\cdot\vec{n}\,ds = \frac{3\pi}{4}.$$

12.



$$\bar{\gamma}_1: \begin{cases} x = 0 \\ y = t \end{cases} \quad 0 \leq t \leq d$$

$$\gamma_2: \begin{cases} x = t \\ y = 0 \end{cases} \quad 0 \leq t \leq 1$$

Pelo teorema da divergência

$$\int_\gamma \vec{F}\cdot\vec{n}\,ds + \int_{\gamma_1} \vec{F}\cdot\vec{n}\,ds + \int_{\gamma_2} \vec{F}\cdot\vec{n}\,ds = 0$$

pois div $\vec{F} = 0$. Temos $\vec{F}(0,t) = \vec{0}$,

logo, o fluxo através de γ_1 é zero. Temos, ainda,

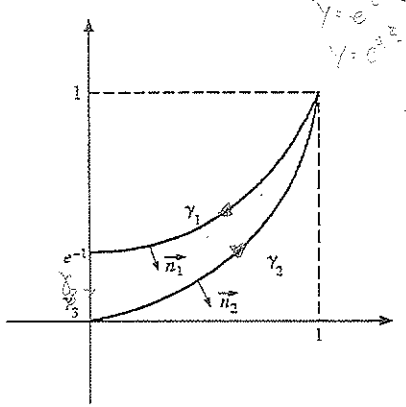
$$\int_{\gamma_2} \vec{F}\cdot\vec{n}\,ds = \int_0^1 \vec{F}(t,0)\cdot(-\vec{j})\,dt = -\frac{3}{2}.$$

Então,

$$\int_\gamma \vec{F}\cdot\vec{n}\,ds = \frac{3}{2}.$$

(Observe que a normal unitária para fora sobre γ_2 é $-\vec{j}$ e sobre γ_1 é $-\vec{i}$.)

13.



$\gamma_3 : \begin{cases} x = 0 \\ y = t \end{cases} \qquad 0 \leq t \leq e^{-1}.$

Como $\vec{F}(0, y) = \vec{0}$, o fluxo através de γ_3 é nulo e, tendo em vista que div $\vec{F} = 0$, pelo teorema da divergência aplicado ao compacto K limitado pelas curvas γ_1, γ_2 e γ_3, temos

$$-\int_{\gamma_1} \vec{F} \cdot \vec{n}_1 \, ds + \int_{\gamma_2} \vec{F} \cdot \vec{n}_2 \, ds = 0.$$

Daí, $\int_{\gamma_1} \vec{F} \cdot \vec{n}_1 \, ds = \int_{\gamma_2} \vec{F} \cdot \vec{n}_2 \, ds$. Temos

$\int_{\gamma_2} \vec{F} \cdot \vec{n}_2 \, ds = \int_0^1 (t^3\vec{i} - 3t^4\vec{j}) \cdot (2t\vec{i} - \vec{j}) \, dt = 1$. Então,

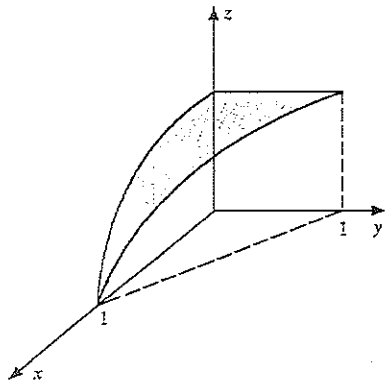
$\int_{\gamma_2} \vec{F} \cdot \vec{n}_2 \, ds = 1$. Ou seja, o fluxo através de γ_1, na direção $\vec{n}_1$, é igual ao fluxo através de γ_2, na direção $\vec{n}_2$.

Observação. Para concluir que as imagens de γ_1 e γ_2 são como na figura anterior, basta mostrar que, para $0 < t < 1$, $g(t) = e^{t^2 - 1} - t^2 > 0$. Para isto, verifique que $g'(t) < 0$, $0 < t < 1$, e que $g(1) = 0$.

CAPÍTULO 9

Exercícios 9.1

1. *g*) Seja $\sigma(u, v) = (u, v, 1 - u^2)$, $u \geq 0$, $v \geq 0$ e $u + v \leq 1$



$$\left.\begin{array}{l} x(u, v) = u \\ y(u, v) = v \\ z(u, v) = 1 - u^2 \end{array}\right\} z = 1 - x^2,\ x \geq 0,\ y \geq 0 \text{ e } x + y \leq 1$$

A imagem de σ coincide com parte do gráfico do cilindro parabólico $z = 1 - x^2$, $x \geq 0$, $y \geq 0$ e $x + y \leq 1$.

4. *d*) $\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 2x\}$
$x^2 - 2x + 1 + y^2 = 1 \Leftrightarrow (x - 1)^2 + y^2 = 1$.
$x - 1 = \cos u$, $y = \operatorname{sen} u$ e $z = v$.

Então, $(x, y, z) = (1 + \cos u, \operatorname{sen} u, v)$, $0 \leq u \leq 2\pi$, $v \in \mathbb{R}$.

e) $(x, y, z) = (v \cos u, v \operatorname{sen} u, \frac{1}{v})$, $0 \leq u \leq 2\pi$, $v > 0$.

Observe que, para cada v fixo, com u variando de 0 a 2π, o ponto (x, y, z) descreve uma circunferência de centro no eixo z, raio v e na altura $z = \frac{1}{v}$. Agora, para cada u fixo,

seja $0v$ o semi-eixo descrito pelo ponto $(v\cos u, v\operatorname{sen} u)$, $v \geqslant 0$. Este semi-eixo $0v$ é obtido, girando o semi-eixo positivo $0x$ de u radianos. Assim, para cada u fixo, com v variando no intervalo $]0, \infty[$, o ponto (x, y, z), descreve o gráfico da função $z = \frac{1}{v}$ no sistema de coordenadas determinado pelos eixos $0v$ e $0z$.

f) $(x, y, z) = (v \cos u, v \operatorname{sen} u, v - v^2)$, $0 \leqslant u \leqslant 2\pi$, $0 \leqslant v \leqslant 1$.

Observação. De modo geral, a superfície obtida pela rotação, em torno do eixo $0z$, da curva $y = 0$ e $z = f(x)$, $a \leqslant x \leqslant b$, com $a > 0$, é dada por $(x, y, z) = (v \cos u, v \operatorname{sen} u, f(v))$, $0 \leqslant u \leqslant 2\pi$ e $a \leqslant v \leqslant b$.

Exercícios 9.2

1. *b*) Seja a superfície $\sigma(u, v) = (\cos u, \operatorname{sen} u, v)$.

Equação do plano tangente no ponto $\sigma\left(\frac{\pi}{2}, 1\right)$:

$(x, y, z) = (0, 1, 1) + s(-1, 0, 0) + t(0, 0, 1), (s, t) \in \mathbb{R}^2$.

c) Seja a superfície $\sigma(u, v)$ e $(2u, + v, u - v, 3u + 2v)$.
Equação do plano tangente no ponto $\sigma(0, 0)$:

$(x, y, z) = s(2, 1, 3) + t(1, -1, 2), (s, t) \in \mathbb{R}^2$.

d) Seja a superfície $\sigma(u, v) = (u - v, u^2 + v^2, uv)$.
Equação do plano tangente no ponto $\sigma(1, 1)$:

$(x, y, z) = (0, 2, 1) + s(1, 2, 1) + t(-1, 2, 1), (s, t) \in \mathbb{R}^2$.

2. Temos $\Gamma'(t) = \frac{\partial \sigma}{\partial u}(\gamma(t))\, u'(t) + \frac{\partial \sigma}{\partial v}(\gamma(t))\, v'(t)$.

De $\frac{\partial \sigma}{\partial u}(\gamma(t)) \wedge \frac{\partial \sigma}{\partial v}(\gamma(t))$ ortogonal a $\frac{\partial \sigma}{\partial u}(\gamma(t))$ e a $\frac{\partial \sigma}{\partial v}(\gamma(t))$ segue que $\frac{\partial \sigma}{\partial u}(\gamma(t)) \wedge \frac{\partial \sigma}{\partial v}(\gamma(t))$ é ortogonal a $\Gamma'(t)$. Pensando geometricamente: $\Gamma'(t)$ é tangente a σ em $\sigma(\gamma(t))$, pois, a imagem de Γ está contida em σ, logo, $\Gamma'(t)$ é ortogonal a $\frac{\partial \sigma}{\partial u}(\gamma(t)) \wedge \frac{\partial \sigma}{\partial v}(\gamma(t))$.

3. Seja $\sigma(u, v) = (x(u, v), y(u, v), t(u, v))$.

Temos

$$\frac{\partial \sigma}{\partial u} \wedge \frac{\partial \sigma}{\partial v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix} =$$

$$= \left(\frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial y}{\partial v}\frac{\partial z}{\partial u}\right)\vec{i} + \left(\frac{\partial z}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial z}{\partial v}\frac{\partial x}{\partial u}\right)\vec{j} +$$

$$+ \left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right)\vec{k} =$$

$$= \begin{vmatrix} \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{vmatrix}\vec{i} + \begin{vmatrix} \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \end{vmatrix}\vec{j} + \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}\vec{k}$$

Portanto,

$$\frac{\partial \sigma}{\partial u} \wedge \frac{\partial \sigma}{\partial v} = \frac{\partial(y, z)}{\partial(u, v)}\vec{i} + \frac{\partial(z, x)}{\partial(u, v)}\vec{j} + \frac{\partial(x, y)}{\partial(u, v)}\vec{k}.$$

Exercícios 9.3

1. *c*) $\sigma(u, v) = (u, v, u^2 + v^2)$, $u^2 + v^2 \leqslant 4$

Temos

$\frac{\partial \sigma}{\partial u} = (1, 0, 2u)$ e $\frac{\partial \sigma}{\partial v} = (1, 0, 2v)$.

$$\frac{\partial \sigma}{\partial u} \wedge \frac{\partial \sigma}{\partial v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 2u \\ 0 & 1 & 2v \end{vmatrix} = -2u\vec{i} - 2v\vec{j} + \vec{k}$$

área de $\sigma = \iint_k \sqrt{4u^2 + 4v^2 + 1}\, du\, dv =$

$$= \int_0^{2\pi}\int_0^2 (4\rho^2 + 1)^{\frac{1}{2}} \rho\, d\rho\, d\theta = \frac{1}{12}\int_0^{2\pi}\left[(4\rho^2 + 1)^{\frac{3}{2}}\right]_0^2 d\theta =$$

$$= \frac{1}{12}\int_0^{2\pi}(17\sqrt{17} - 1)\, d\theta = \frac{\pi}{6}(17\sqrt{17} - 1).$$

d) área de $\sigma = \iint_K \left\| \frac{\partial \sigma}{\partial y} \wedge \frac{\partial \sigma}{\partial v} \right\| du\, dv = \iint_K \sqrt{1 + 4u^2 + 4v^2}\; du\, dv$

Passando para coordenadas polares, temos

área de $\sigma = \int_0^{\pi} \int_0^{e^{-\theta}} \sqrt{1 + 4\rho^2}\; \rho\, d\rho\, d\theta =$

$$= \frac{1}{12} \int_0^{\pi} \left[(1 + 4\rho^2)^{\frac{3}{2}} \right]_0^{e^{-\theta}} d\theta = \frac{1}{12} \int_0^{\pi} (4e^{-2\theta} + 1)^{\frac{3}{2}}\, d\theta - \frac{1}{12} \int_0^{\pi} d\theta.$$

Façamos $u^2 = 4e^{-2\theta} + 1$; $2u\, du = -8e^{-2\theta}\, d\theta$, ou seja, $d\theta = \frac{u\, du}{1 - u^2}$.

Então,

$$\int_0^{\pi} (4e^{-2\theta} + 1)^{\frac{3}{2}}\, d\theta = \int_{\sqrt{5}}^{\sqrt{4e^{-2\pi} + 1}} (u^2)^{\frac{3}{2}}\, \frac{u\, du}{1 - u^2} = \int_{\sqrt{5}}^{\sqrt{4e^{-2\pi} + 1}} \frac{u^4}{1 - u^2}\, du =$$

$$= \int_{\sqrt{5}}^{\sqrt{4e^{-2\pi} + 1}} \left[(u^2 + 1) - \frac{1}{2(u+1)} + \frac{1}{2(u-1)} \right] du =$$

$$= -\left[\frac{u^3}{3} + u - \frac{1}{2} \ln (u+1) + \frac{1}{2} \ln (u-1) \right]_{\sqrt{5}}^{\sqrt{4e^{-2\pi} + 1}}$$

área de $\sigma = \frac{1}{12} \left[\frac{u^3}{3} + u - \frac{1}{2} \ln \frac{u-1}{u+1} \right]_{\sqrt{1 + 4e^{-2\pi}}}^{\sqrt{5}} - \frac{\pi}{12}$.

2. Uma parametrização para superfície gerada pela rotação em torno do eixo 0z do conjunto A é (veja Exercício 2 da Seção 9.1):
$(x, y, z) = ((2 + \cos v) \operatorname{sen} u, (2 + \cos v) \cos u, \operatorname{sen} v)$, $0 \leq u \leq 2\pi$, $0 \leq v \leq 2\pi$.

Temos

$$\frac{\partial \sigma}{\partial u} \wedge \frac{\partial \sigma}{\partial v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ (2 + \cos v) \cos u & (2 + \cos v)(-\operatorname{sen} u) & 0 \\ -\operatorname{sen} v \operatorname{sen} u & -\operatorname{sen} v \cos u & \cos v \end{vmatrix} =$$

$$= (2 + \cos v) \cos v\, (-\operatorname{sen} u)\, \vec{i} - (2 + \cos v) \cos u \cos v\, \vec{j} - (2 + \cos v) \operatorname{sen} v\, \vec{k}.$$

área de $\sigma = \iint_K \left\| \frac{\partial \sigma}{\partial u} \wedge \frac{\partial \sigma}{\partial v} \right\| du\, dv =$

$$= \int_0^{2\pi} \int_0^{2\pi} \sqrt{(2 + \cos v)^2 \cos^2 v \operatorname{sen}^2 u + (2 + \cos v)^2 \operatorname{sen}^2 v + (2 + \cos v)^2 \cos^2 u \cos^2 v}\; du\, dv =$$

$$= \int_0^{2\pi} \left[\int_0^{2\pi} (2 + \cos v)\, du \right] dv = 8\pi^2.$$

4. Seja $z^2 = 4 - x^2$, $z \geq 0$. Então, $z = \sqrt{4 - x^2}$, $\frac{\partial z}{\partial x} = -\frac{x}{\sqrt{4 - x^2}}$ e $\frac{\partial z}{\partial y} = 0$.

Observe que o círculo $x^2 + y^2 \leq 4$ está contido no domínio de $z = \sqrt{4 - x^2}$.

Área $= \iint_K \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\; dx\, dy$, onde K é o círculo $x^2 + y^2 \leq 4$.

Área $= 2 \int_{-2}^{2} \int_0^{\sqrt{4 - x^2}} \sqrt{1 + \left(-\frac{x}{\sqrt{4 - x^2}} \right)^2}\; dy\, dx =$

$$= 4 \int_{-2}^{2} \int_0^{\sqrt{4 - x^2}} \frac{dy}{\sqrt{4 - x^2}}\, dy\, dx = 4 \int_{-2}^{2} \frac{1}{\sqrt{4 - x^2}} [y]_0^{\sqrt{4 - x^2}}\; dx = 16.$$

5. Área $= \iint_K \frac{dx\, dy}{\sqrt{1 - x^2 - y^2}}$, onde $K = \{(x, y) \mid \underbrace{1 - x^2 - y^2}_{z^2} \geq x^2 + y^2\}$.

Passando para coordenadas polares, vem área $= \int_0^{2\pi} \int_0^{\frac{\sqrt{2}}{2}} \frac{\rho\, d\rho\, d\theta}{\sqrt{1 - \rho^2}} = \pi (2 - \sqrt{2})$.

7. De $x^2 + y^2 + z^2 = 2$ e $z = x^2 + y^2$ segue $z + z^2 = 2$ e, portanto, $z = 1$, pois, $z \geq 0$. Assim, a interseção das superfícies é a circunferência $x^2 + y^2 = 1$, $z = 1$. Queremos então a área da superfície

$$z = \sqrt{2 - x^2 - y^2},\ x^2 + y^2 \leq 1.$$

Área $= \iint_K \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\; dxdy$

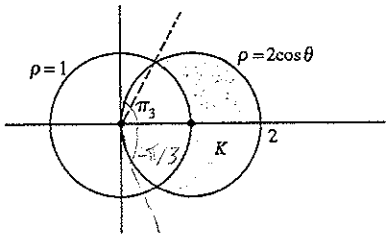
onde K é o círculo $x^2 + y^2 \leq 1$. Temos

área $= \iint_K \sqrt{\frac{2}{2 - x^2 - y^2}}\; dxdy.$

Passando para coordenadas polares,

$$\text{área} = \int_0^{2\pi}\int_0^1 \frac{\sqrt{2}\,\rho}{\sqrt{2-\rho^2}}\,d\rho d\theta = 2\pi\sqrt{2}\int_0^1 \frac{\rho}{\sqrt{2-\rho^2}}\,d\rho = \frac{4\pi}{2+\sqrt{2}}.$$

8.

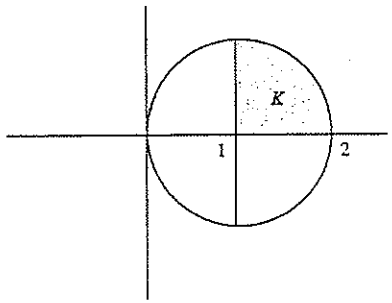


$$\text{Área} = \iint_K \sqrt{1+\frac{x^2}{x^2+y^2}+\frac{y^2}{x^2+y^2}}\,dxdy = \sqrt{2}\iint_K dxdy$$

onde K é a região hachurada. Em coordenadas polares, temos

$$\text{área} = 2\sqrt{2}\int_0^{\frac{\pi}{3}}\int_1^{2\cos\theta}\rho\,d\rho\,d\theta = \frac{\sqrt{6}}{2}+\frac{\pi\sqrt{2}}{3}.$$

10.



$$\text{Área} = \iint_K \sqrt{1+\frac{x^2}{x^2+y^2}+\frac{y^2}{x^2+y^2}}\,dx\,dy = \sqrt{2}\iint_K dx\,dy = \frac{\pi\sqrt{2}}{4}$$

onde K é a região hachurada.

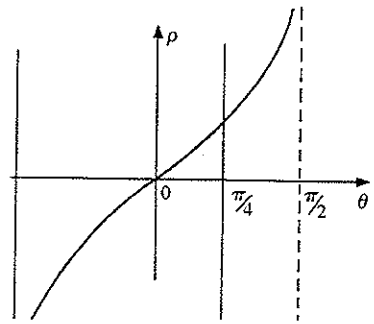
12. Seja $z = xy$. Temos $\frac{\partial z}{\partial x} = y$ e $\frac{\partial z}{\partial y} = x$.

$$\text{Área} = \iint_K \sqrt{1+x^2+y^2}\,dx\,dy, \text{ onde } K = \{(x,y)\,|\,1 \leqslant x^2+y^2 \leqslant 4\}.$$

Passando para coordenadas polares,

$$\text{área} = \int_0^{2\pi}\int_1^2 \sqrt{1+\rho^2}\;\rho\,d\rho\,d\theta = \frac{2\pi}{3}(5\sqrt{5}-2\sqrt{2}).$$

13. Seja $z = xy$.



$$\text{Área} = \iint_K \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dx\,dy =$$

$$= \iint_K \sqrt{1+x^2+y^2}\,dx\,dy = \int_0^{\frac{\pi}{4}}\int_0^{\operatorname{tg}\theta}\left(\sqrt{1+\rho^2}\right)\rho\,d\rho\,d\theta =$$

$$= \frac{1}{2}\cdot\frac{2}{3}\int_0^{\frac{\pi}{4}}\left[(1+\rho^2)^{\frac{3}{2}}\right]_0^{\operatorname{tg}\theta} d\theta = \frac{1}{3}\int_0^{\frac{\pi}{4}}[(1+\operatorname{tg}^2\theta)^{\frac{3}{2}}-1]\,d\theta =$$

$$= \frac{1}{3}\int_0^{\frac{\pi}{4}}\sec^3\theta\,d\theta - \frac{1}{3}\int_0^{\frac{\pi}{4}} d\theta = \frac{1}{3}\left[\frac{1}{2}\sec\theta\operatorname{tg}\theta+\frac{1}{2}\ln|\sec\theta+\operatorname{tg}\theta|\right]_0^{\frac{\pi}{4}} -$$

$$-\frac{1}{3}\cdot\frac{\pi}{4} = \frac{1}{3}\left[\frac{\sqrt{2}}{2}+\frac{\ln(\sqrt{2}+1)}{2}\right]-\frac{\pi}{12}.$$

14.

$$\text{Área} = \iint_K \sqrt{1+x^2+y^2}\,dx\,dy =$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_0^{\sqrt{\cos 2\theta}} (\sqrt{1+\rho^2})\, \rho\, d\rho\, d\theta = \frac{1}{2}\cdot\frac{2}{3}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left[(1+\rho^2)^{\frac{3}{2}}\right]_0^{\sqrt{\cos 2\theta}} d\theta =$$

$$= \frac{1}{3}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (1+\cos 2\theta)^{\frac{3}{2}}\, d\theta - \frac{1}{3}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} d\theta =$$

$$= \frac{1}{3}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (2\cos^2\theta)^{\frac{3}{2}}\, d\theta - \frac{1}{3}[\theta]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = \frac{2\sqrt{2}}{3}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos^3\theta\, d\theta - \frac{\pi}{6} = \frac{10}{9} - \frac{\pi}{6}.$$

15. Área $= \iint_K \sqrt{1+4x^2+16y^2}\, dx\, dy$, onde

$K = \{(x, y) \mid 4x^2 + 16y^2 \leq 1\}$

Façamos a mudança para coordenadas polares:

$$\begin{cases} 2x = \rho\cos\theta \\ 4y = \rho\,\text{sen}\,\theta \end{cases} \qquad 0 \leq \theta \leq 2\pi \text{ e } 0 \leq \rho \leq 1.$$

$$dx\, dy = \left|\frac{\sigma(x, y)}{\sigma(\rho, \theta)}\right| d\rho\, d\theta = \frac{\rho}{8}\, d\rho\, d\theta.$$

Portanto,

$$\text{área} = \int_0^{2\pi}\left(\int_0^1 \sqrt{1+\rho^2}\,\frac{\rho}{8}\, d\rho\right) d\theta = \frac{\pi}{12}(2\sqrt{2}-1).$$

Exercícios 9.4

1. *c*) $f(x, y, z) = x^2 + y^2$ e $\sigma(u, v) = (u, v, u^2 + v^2)$, $u^2 + v^2 \leq 1$.

Temos, $f(\sigma(u, v)) = u^2 + v^2$, $\dfrac{\partial\sigma}{\partial u} = (1, 0, 2u)$ e $\dfrac{\partial\sigma}{\partial v} = (0, 1, 2v)$.

$$\frac{\partial\sigma}{\partial u} \wedge \frac{\partial\sigma}{\partial v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 2u \\ 0 & 1 & 2v \end{vmatrix} = -2u\vec{i} - 2v\vec{j} + \vec{k}.$$

$$\iint_\sigma f(x, y, z)\, dS = \iint_K f(\sigma(u, v)) \left\| \frac{\partial\sigma}{\partial u} \wedge \frac{\partial\sigma}{\partial v} \right\| du\, dv =$$

$$= \iint_K (u^2 + v^2)\sqrt{1+4u^2+4v^2}\, du\, dv$$

Passando para coordenadas polares,

$$\int_0^{2\pi}\int_0^1 \rho^2\sqrt{1+4\rho^2}\, \rho\, d\rho\, d\theta = \int_0^{2\pi}\left[\int_0^1 (1+4\rho^2)^{\frac{1}{2}}\rho^3\, d\rho\right] d\theta =$$

(por partes)

$$= \frac{1}{8}\int_0^{2\pi}\left[\frac{2}{3}\rho^2(1+4\rho^2)^{\frac{3}{2}} - \frac{1}{15}(1+4\rho^2)^{\frac{5}{2}}\right]_0^1 d\theta = \frac{\pi}{4}\left(\frac{5\sqrt{5}}{3} + \frac{1}{15}\right).$$

g) $f(x, y, z) = x$, $\sigma(u, v) = (u, v, \sqrt{u^2 + v^2})$, $1 \leq u^2 + v^2 \leq 9$, e $f(\sigma(u, v)) = u$. Temos

$\dfrac{\partial\sigma}{\partial u} = \left(1, 0, \dfrac{u}{\sqrt{u^2+v^2}}\right)$ e $\dfrac{\partial\sigma}{\partial v} = \left(0, 1, \dfrac{v}{\sqrt{u^2+v^2}}\right)$. Daí,

$$\frac{\partial\sigma}{\partial u} \wedge \frac{\partial\sigma}{\partial v} = -\frac{u\,\vec{i}}{\sqrt{u^2+v^2}} - \frac{v\,\vec{j}}{\sqrt{u^2+v^2}} + \vec{k} \text{ e}$$

$\left\| \dfrac{\partial\sigma}{\partial u} \wedge \dfrac{\partial\sigma}{\partial v} \right\| = \sqrt{\dfrac{u^2}{u^2+v^2} + \dfrac{v^2}{u^2+v^2} + 1} = \sqrt{2}$. Então,

$$\iint_\sigma f(x, y, z)\, ds = \iint_K f(\sigma(u, v)) \left\| \frac{\partial\sigma}{\partial u} \wedge \frac{\partial\sigma}{\partial v} \right\| du\, dv =$$

$$= \iint_K u \cdot \sqrt{2} \cdot du\, dv = \sqrt{2}\int_1^3\int_0^{2\pi} \rho^2 \cos\theta\, d\theta\, d\rho = 0.$$

h) $f(x, y, z) = z$, $\sigma(u, v) = (u, v, \sqrt{u^2+v^2})$ e $f(\sigma(u, v)) = \sqrt{u^2+v^2}$.

Tendo em vista o item *g*,

$$\left\| \frac{\partial\sigma}{\partial u} \wedge \frac{\partial\sigma}{\partial v} \right\| = \sqrt{2}.$$

Então,

$$\iint_\sigma f(x, y, z)\, dS = \iint_K f(\sigma(u, v)) \left\| \frac{\partial\sigma}{\partial u} \wedge \frac{\partial\sigma}{\partial v} \right\| du\, dv = \iint_K \sqrt{u^2+v^2} \cdot \sqrt{2} \cdot dx\, dv$$

onde σ é a parte do cone $z^2 = x^2 + y^2$ que se encontra acima do parabolóide $4z = x^2 + y^2 + 3$.

Devemos ter, $z^2 - 4z + 3 \leq 0$ e, portanto, $1 \leq z \leq 3$.

(Veja: $z = \sqrt{x^2+y^2}$ e $z = \dfrac{x^2+y^2+3}{4}$; devemos ter então $\sqrt{x^2+y^2} \geqslant \dfrac{x^2+y^2+3}{4}$, ou seja, $z \geqslant \dfrac{z^2+3}{4}$ e, portanto, $z^2 - 4z + 3 \leqslant 0$. A região de integração é a coroa circular de raios 1 e 3 e com centro na origem.)

Em coordenadas polares,

$$\iint_\sigma f(x, y, z)\, dS = \int_0^{2\pi}\int_1^3 \sqrt{2}\,\rho^2\, d\rho\, d\theta = \frac{52\sqrt{2}\,\pi}{3}.$$

i) $f(x, y, z) = \sqrt{1-x^2}$, $\sigma(u, v) = (u, v, \sqrt{1-u^2})$ e $f(\sigma(u, v)) = \sqrt{1-u^2}$.

Temos

$$\frac{\partial\sigma}{\partial u}\left(1, 0, -\frac{u}{\sqrt{1-u^2}}\right), \frac{\partial\sigma}{\partial v} = (0, 1, 0),$$

$$\frac{\partial\sigma}{\partial u} \wedge \frac{\partial\sigma}{\partial v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -\dfrac{u}{\sqrt{1-u^2}} \\ 0 & 1 & 0 \end{vmatrix} = \frac{-u}{\sqrt{1-u^2}}\vec{i} + \vec{k} \text{ e}$$

$$\left\| \frac{\partial\sigma}{\partial u} \wedge \frac{\partial\sigma}{\partial v} \right\| = \sqrt{1+\frac{u^2}{1-u^2}} = \frac{1}{\sqrt{1-u^2}}.$$

$$\iint_\sigma f(x, y, z)\, dS = \iint_K f(\sigma(u, v)) \left\| \frac{\partial\sigma}{\partial u} \wedge \frac{\partial\sigma}{\partial v} \right\| du\, dv =$$

$$= \iint_K \sqrt{1-u^2} \cdot \frac{1}{\sqrt{1-u^2}}\, du\, dv = \iint_K du\, dv, \text{ onde}$$

K é a região elíptica $2u^2 + v^2 \leqslant 1$, cuja área é $\dfrac{1}{\sqrt{2}} \cdot 1 \cdot \pi = \dfrac{\pi}{\sqrt{2}}$.

Então,

$$\iint_\sigma f(x, y, z)\, dS = \iint_K du\, dv = \frac{\sqrt{2}}{2}\pi.$$

(Veja: $z^2 + x^2 = 1 \Leftrightarrow z^2 = 1 - x^2$; $z = \sqrt{x^2+y^2} \Leftrightarrow z^2 = x^2 + y^2$ $(z \geqslant 0)$; devemos ter então $1 - x^2 \geqslant x^2 + y^2$ e, portanto, $2x^2 + y^2 \leqslant 1$.)

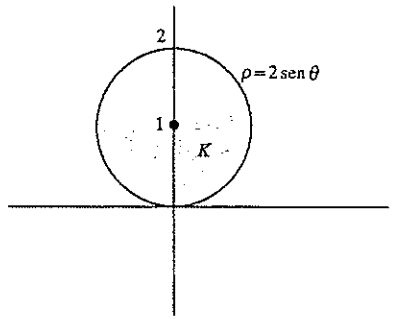
j) $f(x, y, z) = \dfrac{z}{\sqrt{1+4x^2+4y^2}}$; $\sigma(u, v) = (u, v, 1-u^2-v^2)$ e $f(\sigma(u, v)) = \dfrac{1-u^2-v^2}{\sqrt{1+4u^2+4v^2}}$.

Temos

$$\frac{\partial\sigma}{\partial u} \wedge \frac{\partial\sigma}{\partial v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -2u \\ 0 & 1 & -2v \end{vmatrix} = 2u\vec{i} + 2v\vec{j} + \vec{k} \text{ e} \left\| \frac{\partial\sigma}{\partial u} \wedge \frac{\partial\sigma}{\partial v} \right\| = \sqrt{1+4u^2+4v^2}.$$

$$\iint_\sigma f(x, y, z)\, dS = \iint_K f(\sigma(u, v)) \left\| \frac{\partial\sigma}{\partial u} \wedge \frac{\partial\sigma}{\partial v} \right\| du\, dv = \iint_K (1-u^2-v^2)\, du\, dv,$$

onde $K = \{(u, v) \mid u^2 + v^2 \leqslant 2v\}$.



$$\iint_\sigma f(x, y, z)\, dS = \int_0^\pi \left[\int_0^{2\text{sen}\theta} (1-\rho^2)\,\rho\, d\rho \right] d\theta = -\frac{\pi}{2}.$$

2. *a)* $M = \iint_\sigma f(x, y, z)\, dS = \iint_K f(\sigma(u, v)) \left\| \dfrac{\partial\sigma}{\partial u} \wedge \dfrac{\partial\sigma}{\partial v} \right\| du\, dv =$

$$= \iint_K \sqrt{1+4u^2+4v^2}\, du\, dv.$$

Passando para coordenadas polares, temos

$$M = \int_0^{2\pi}\int_0^1 \sqrt{1+4\rho^2}\,\rho\, d\rho\, d\theta = \frac{\pi}{6}(5\sqrt{5}-1).$$

$$x_c = \frac{\iint_\sigma x\, dm}{M}; y_c = \frac{\iint_\sigma y\, dm}{M}; z_c = \frac{\iint_\sigma z\, dm}{M}$$

$$\iint_\sigma x\, dm = \int_0^{2\pi}\int_0^1 (\rho\cos\theta)\sqrt{1+4\rho^2}\,\rho\, d\rho\, d\theta = 0,$$

$$\iint_\sigma y\, dm = \int_0^{2\pi}\int_0^1 (\rho\,\text{sen}\,\theta)\sqrt{1+4\rho^2}\,\rho\, d\rho\, d\theta = 0 \text{ e}$$

$$\iint_\sigma z\,dm = \iint_K (u^2+v^2)\sqrt{1+4u^2+4v^2}\;du\,dv =$$

$$= \int_0^{2\pi}\int_0^1 \rho^2\sqrt{1+4\rho^2}\;\rho\,d\rho\,d\theta = \frac{\pi}{4}\left(\frac{5\sqrt{5}}{3}+\frac{1}{15}\right)$$ (veja Exercício 1, *c*, desta seção).

Centro de massa: $\left(0, 0, \dfrac{25\sqrt{5}+1}{10(5\sqrt{5}-1)}\right)$.

3. *a*)

$\sigma(u, v) = (u, v, \sqrt{1-u^2-v^2})$, $\dfrac{\partial\sigma}{\partial u} = (1, 0, -\dfrac{u}{\sqrt{1-u^2-v^2}})$ e $\dfrac{\partial\sigma}{\partial v} = (0, 1, -\dfrac{v}{\sqrt{1-u^2-v^2}})$.

$$\frac{\partial\sigma}{\partial u}\wedge\frac{\partial\sigma}{\partial v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -\dfrac{u}{\sqrt{1-u^2-v^2}} \\ 0 & 1 & -\dfrac{v}{\sqrt{1-u^2-v^2}} \end{vmatrix} = \frac{u}{\sqrt{1-u^2-v^2}}\vec{i} + \frac{v}{\sqrt{1-u^2-v^2}}\vec{j} + \vec{k}.$$

$$\left\|\frac{\partial\sigma}{\partial u}\wedge\frac{\partial\sigma}{\partial v}\right\| = \sqrt{1+\frac{u^2}{1-u^2-v^2}+\frac{v^2}{1-u^2-v^2}} = \frac{1}{\sqrt{1-u^2-v^2}}.$$

$$M = \iint_\sigma f(x, y, z)\,dS = \iint_K f(\sigma(u, v))\left\|\frac{\partial\sigma}{\partial u}\wedge\frac{\partial\sigma}{\partial v}\right\| du\,dv =$$

$$= \iint_K \sqrt{1-u^2-v^2}\cdot\frac{1}{\sqrt{1-u^2-v^2}}\,du\,dv = \int_0^{2\pi}\int_0^1 \rho\,d\rho\,d\theta = \pi.$$

c) $f(x, y, z) = 2$ e $\sigma(u, v) = (u, v, 1-u^2)$, $0 \leq u \leq 1$, $0 \leq v \leq 1$.

$$\frac{\partial\sigma}{\partial u}\wedge\frac{\partial\sigma}{\partial v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -2u \\ 0 & 1 & 0 \end{vmatrix} = 2u\vec{i} + \vec{k}$$ e $$\left\|\frac{\partial\sigma}{\partial u}\wedge\frac{\partial\sigma}{\partial v}\right\| = \sqrt{1+4u^2}.$$

$$M = \int_0^1\int_0^1 2\sqrt{1+4u^2}\;du\,dv = 2\int_0^1\left[\int_0^1\sqrt{1+4u^2}\;du\right]dv = \sqrt{5}+\frac{1}{2}\ln(2+\sqrt{5}).$$

4. $\sigma:\begin{cases} x = R\cos\theta\,\operatorname{sen}\varphi \\ y = R\operatorname{sen}\theta\,\operatorname{sen}\varphi \\ z = R\cos\varphi \end{cases}$ $0 \leq \theta \leq 2\pi$ e $-\dfrac{\pi}{2} \leq \varphi \leq \dfrac{\pi}{2}$ é uma parametrização

para a superfície esférica $x^2+y^2+z^2 = R^2$.

Temos $\left\|\dfrac{\partial\sigma}{\partial\varphi}\wedge\dfrac{\partial\sigma}{\partial\theta}\right\| = \sqrt{R^4\operatorname{sen}^2\varphi}$, ou seja, $\left\|\dfrac{\partial\sigma}{\partial\varphi}\wedge\dfrac{\partial\sigma}{\partial\theta}\right\| = R^2|\operatorname{sen}\varphi|$.

(Confira.) Temos

$$I = k\int_0^{2\pi}\int_{-\pi/4}^{\pi/4}(x^2+y^2)\left\|\frac{\partial\sigma}{\partial\varphi}\wedge\frac{\partial\sigma}{\partial\theta}\right\| d\varphi\,d\theta = 2k\pi R^2\int_{-\pi/4}^{\pi/4}\operatorname{sen}^2\varphi\,|\operatorname{sen}\varphi|\,d\varphi.$$

Como o integrando é função par, $I = 4k\pi R^4\int_0^{\pi/4}\operatorname{sen}^3\varphi\,d\varphi = \dfrac{8k\pi R^4}{3}$, pois,

$\int_0^{\pi/2}\operatorname{sen}^3\theta\,d\theta = \dfrac{2}{3}$. Sendo $4\pi R^2$ a área da superfície esférica e k a densidade, a massa será $M = 4k\pi R^2$ e, portanto, $I = \dfrac{2MR^2}{3}$.

5. $\sigma(u, v) = (u, v, u^2+v^2)$, $u^2+v^2 \leq R^2$.

$$\frac{\partial\sigma}{\partial u}\wedge\frac{\partial\sigma}{\partial v} = -2u\vec{i} - 2v\vec{j} + \vec{k}.$$

$I = k\iint_K (u^2+v^2)\sqrt{1+4u^2+4v^2}\;du\,dv$, onde K é o círculo $u^2+v^2 \leq R^2$ e k a densidade. Fazendo, $2u = \rho\cos\theta$ e $2v = \rho\operatorname{sen}\theta$, teremos $dudv = \dfrac{1}{4}\rho\,d\rho\,d\theta$, $0 \leq \theta \leq 2\pi$ e $0 \leq \rho \leq 2R$. Então, $I = \dfrac{k}{16}\int_0^{2\pi}\int_0^{2R}\rho^3\sqrt{1+\rho^2}\;d\rho\,d\theta$ e daí

$I = \dfrac{k\pi}{8}\int_0^{2R}\rho^3\sqrt{1+\rho^2}\;d\rho$. Fazendo $s = 1+\rho^2$, teremos

$$I = \frac{k\pi}{16}\int_1^{1+4R^2}(1-s)\sqrt{s}\;ds = \frac{k\pi}{8}\left\{5\left[\sqrt{(1+4R^2)^3}-1\right]-3\left[\sqrt{(1+4R^2)^5}-1\right]\right\}.$$

Por outro lado, área de $\sigma = \iint_K\sqrt{1+4u^2+4v^2}\;dudv = \dfrac{1}{4}\int_0^{2\pi}\int_0^{2R}\rho\sqrt{1+\rho^2}\;d\rho$,

ou seja, área de $\sigma = \dfrac{\pi}{6}\left[\sqrt{(1+4R^2)^3}-1\right]$. De $M = k\times$ (área de σ), segue que

$$I = \frac{3M}{4}\left[5-\frac{3\sqrt{(1+4R^2)^5}-3}{\sqrt{(1+4R^2)^3}-1}\right].$$

CAPÍTULO 10

Exercícios 10.1

1. Sejam $B = \{(x, y, z) \in \mathbb{R}^3 \mid x \geqslant 0, y \geqslant 0 \text{ e } 0 \leqslant z \leqslant 1 - x - y\}$, σ a fronteira de B e $\vec{F}(x, y, z) = x\vec{i} + y\vec{j} + z\vec{k}$.

Temos

$\sigma(u, v) = (u, v, 1 - u - v)$ e $\dfrac{\partial\sigma}{\partial u} \wedge \dfrac{\partial\sigma}{\partial v} = \vec{i} + \vec{j} + \vec{k}$.

$$\iint_\sigma \vec{F}\cdot\vec{n}\, dS = \iint_K \vec{F}(\sigma(u,v)) \underbrace{\frac{\dfrac{\partial\sigma}{\partial u}\wedge\dfrac{\partial\sigma}{\partial v}}{\left\|\dfrac{\partial\sigma}{\partial u}\wedge\dfrac{\partial\sigma}{\partial v}\right\|}}_{\vec{n}} \cdot \underbrace{\left\|\frac{\partial\sigma}{\partial u}\wedge\frac{\partial\sigma}{\partial v}\right\| du\, dv}_{dS}$$

ou seja,

$$\iint_\sigma \vec{F}\cdot\vec{n}\, ds = \iint_K (u\vec{i} + v\vec{j} + (1-u-v)\vec{k})\cdot(\vec{i}+\vec{j}+\vec{k})\, du\, dv =$$

$$= \iint_K du\, dv = \frac{1}{2}, \text{ pois } K \text{ é o triângulo } u \geqslant 0,\ v \geqslant 0,\ u + v \leqslant 1.$$

Então, $\iint_\sigma \vec{F}\cdot\vec{n}\, dS = \dfrac{1}{2}$.

Por outro lado,

$$\iiint_B \underbrace{\operatorname{div}\vec{F}}_{3}\, dx\, dy\, dz = 3\iiint_B dx\, dy\, dz =$$

$$= 3\iint_A \left[\int_0^{1-x-y} dz\right] dy\, dx = 3\iint_A (1-x-y)\, dy\, dx, \text{ onde } A \text{ é o}$$

triângulo $x + y \leqslant 1, x \geqslant 0$ e $y \geqslant 0$.

$$\iiint_B \operatorname{div}\vec{F}\, dx\, dy\, dz = 3\int_0^1 \left[\int_0^{1-x} (1-x-y)\, dy\right] dx =$$

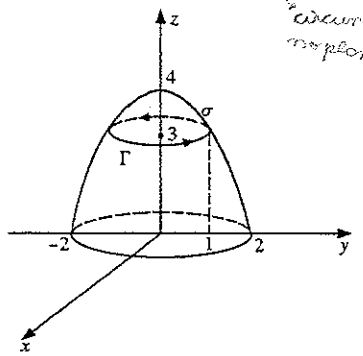
$$= 3\int_0^1 \frac{(1-x)^2}{2}\, dx = \frac{1}{2}.$$

Comparando ① e ② segue:

$$\iint_\sigma \vec{F}\cdot\vec{n}\, dS = \iiint_B \operatorname{div}\vec{F}\, dx\, dy\, dz.$$

2. Sejam $\sigma(u, v) = (u, v, 4 - u^2 - v^2)$, $u^2 + v^2 \leqslant 1$, $\Gamma(t) = (\cos t, \operatorname{sen} t, 3)$, $0 \leqslant t \leqslant 2\pi$ e $\vec{F}(x, y, z) = \vec{i} + (x + y + z)\vec{j}$.

a)



b) Temos:

$$\operatorname{rot}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 1 & x+y+z & 0 \end{vmatrix} = -\vec{i} + \vec{k} \text{ e}$$

$$\frac{\partial\sigma}{\partial u}\wedge\frac{\partial\sigma}{\partial v} = 2u\vec{i} + 2v\vec{j} + \vec{k}.$$

Então,

$$\text{①} \iint_\sigma \operatorname{rot}\vec{F}\cdot\vec{n}\, dS = \iint_K \operatorname{rot} F(\sigma(u,v))\cdot \underbrace{\frac{\dfrac{\partial\sigma}{\partial u}\wedge\dfrac{\partial\sigma}{\partial v}}{\left\|\dfrac{\partial\sigma}{\partial u}\wedge\dfrac{\partial\sigma}{\partial v}\right\|}}_{\vec{n}} \cdot \underbrace{\left\|\frac{\partial\sigma}{\partial u}\wedge\frac{\partial\sigma}{\partial v}\right\| du\, dv}_{dS} =$$

$$= \iint_K (-\vec{i} + \vec{k})(2u\vec{i} + 2v\vec{j} + \vec{k})\, du\, dv = \iint_K (-2u + 1)\, du\, dv =$$

$$= \int_0^{2\pi}\left[\int_0^1 (-2\rho\cos\theta + 1)\,\rho\, d\rho\right] d\theta = \pi.$$

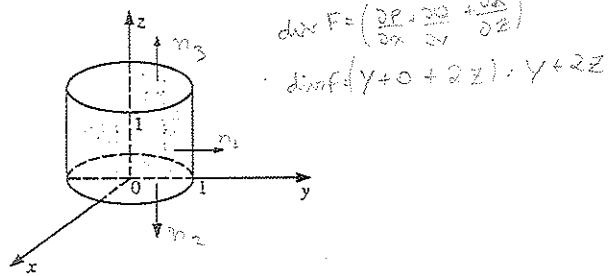
Por outro lado,

$$\textcircled{2}\ \int_\Gamma \vec{F}\,d\Gamma = \int_0^{2\pi}\left[\vec{i} + (\operatorname{sen} t + \cos t + 3)\,\vec{j}\right]\cdot\left[-\operatorname{sen} t\,\vec{i} + \cos t\,\vec{j}\right] dt = \pi.$$

De ① e ② segue

$$\iint_\sigma \operatorname{rot}\vec{F}\cdot\vec{n}\,dS = \int_\Gamma \vec{F}\,d\Gamma.$$

3. Sejam B o cilindro $x^2 + y^2 \leq 1$ e $0 \leq z \leq 1$, e $\vec{F}(x, y, z) = xy\vec{i} - \vec{j} + z^2\vec{k}$.



Consideremos a cadeia $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ assim definida:
$\sigma_1(u, v) = (\cos u, \operatorname{sen} u, v)$, $0 \leq v \leq 1$, $0 \leq u \leq 2\pi$ (superfície lateral do cilindro),
$\sigma_2(u, v) = (u, v, 0)$, $u^2 + v^2 \leq 1$ (base inferior do cilindro) e
$\sigma_3(u, v) = (u, v, 1)$, $u^2 + v^2 \leq 1$ (base superior do cilindro).

Temos

$$\frac{\partial\sigma_1}{\partial u} \wedge \frac{\partial\sigma_1}{\partial v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\operatorname{sen} u & \cos u & 0 \\ 0 & 0 & 1 \end{vmatrix} = \cos u\,\vec{i} + \operatorname{sen} u\,\vec{j}.$$

Seja $\vec{n}_1 = \cos u\,\vec{i} + \operatorname{sen} u\,\vec{j}$ a normal a σ_1.

$$\iint_{\sigma_1} \vec{F}\cdot n\,dS = \iint_K \left[(\cos u \operatorname{sen} u)\,\vec{i} - \vec{j} + v^2\vec{k}\right]\cdot(\cos u\,\vec{i} + \operatorname{sen} u\,\vec{j})\,du\,dv =$$

$$= \iint_K (\cos^2 u \operatorname{sen} u - \operatorname{sen} u)\,du\,dv =$$

$$= \int_0^1\left[-\frac{\cos^3 u}{3} + \cos u\right]_0^{2\pi} dv = 0.$$

Portanto, o fluxo de $\vec{F}$ através da superfície lateral do cilindro é zero.
Seja $\vec{n}_2 = -\vec{k}$ a normal a σ_2 e $\vec{n}_3 = \vec{k}$ a normal a σ_3.

Temos

$$\iint_{\sigma_2} \vec{F}\cdot\vec{n}\,dS = \iint_K F(\sigma_2(u, v))\cdot\vec{n}_2(\sigma_2(u, v))\,du\,dv =$$

$$= \iint_K (uv\vec{i} - v\vec{j})\cdot(-\vec{k})\,du\,dv = 0.$$

$$\iint_{\sigma_3} \vec{F}\cdot\vec{n}\,dS = \iint_K F(\sigma_3(u, v))\cdot\vec{n}_3(\sigma_3(u, v))\,du\,dv =$$

$$= \iint_K (uv\vec{i} - v\vec{j} + \vec{k})\cdot(\vec{k})\,du\,dv = \iint_K du\,dv =$$

$$= \int_0^{2\pi}\int_0^1 \rho\,d\rho\,d\theta = \pi.$$

Então,

$$\iint_\sigma \vec{F}\cdot\vec{n}\,dS = \iint_{\sigma_1} \vec{F}\cdot\vec{n}_1\,dS + \iint_{\sigma_2} \vec{F}\cdot\vec{n}_2\,dS + \iint_{\sigma_3} \vec{F}\cdot\vec{n}_3\,dS =$$

$$= 0 + 0 + \pi \Rightarrow \iint_\sigma \vec{F}\cdot\vec{n}\,dS = \pi \quad \textcircled{1}$$

Por outro lado,

$$\iiint_B \operatorname{div}\vec{F}\,dx\,dy\,dz = \iiint_B (y + 2z)\,dx\,dy\,dz =$$

$$= \iint_K\left[\int_0^1 (y + 2z)\,dz\right]dx\,dy = \iint_K (y + 1)\,dx\,dy$$

onde K é o círculo $x^2 + y^2 \leq 1$.

Mudando para coordenadas polares:

$$\iint_K (y + 1)\,dx\,dy = \int_0^{2\pi}\left[\int_0^1 (\rho\operatorname{sen}\theta + 1)\,\rho\,d\rho\right]d\theta =$$

$$= \int_0^{2\pi}\left[\frac{\rho^3}{3}\operatorname{sen}\theta + \frac{\rho^2}{2}\right]_0^1 d\theta = \frac{1}{3}\int_0^{2\pi}\operatorname{sen}\theta\,d\theta + \frac{1}{2}\int_0^{2\pi} d\theta = \pi \quad \textcircled{2}$$

De ① e ②, segue

$$\iint_\sigma \vec{F}\cdot\vec{n}\,dS = \iiint_B \operatorname{div}\vec{F}\,dx\,dy\,dz.$$

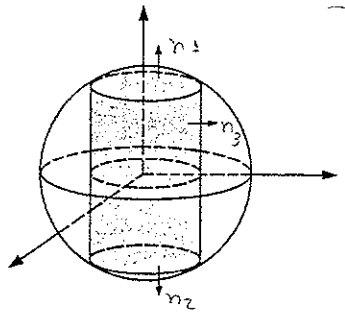
4. Sejam $B = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 \leq 1 \text{ e } x^2 + y^2 + z^2 \leq 4\}$, σ a fronteira de B e
$\vec{F}(x, y, z) = x\vec{i} + y\vec{j} + z\vec{k}$.

Consideremos a cadeia $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ assim definida:

$\sigma_1(u, v) = (u, v, \sqrt{4-u^2-v^2})$, $u^2+v^2 \leq 1$ (calota superior)

$\sigma_2(u, v) = (u, v, -\sqrt{4-u^2-v^2})$, $u^2+v^2 \leq 1$ (calota inferior)

$\sigma_3(u, v) = (\cos u, \operatorname{sen} u, v)\ 0 \leq u \leq 2\pi, -\sqrt{3} \leq v \leq \sqrt{3}$ (superfície lateral do cilindro)



Temos

$$\iint_\sigma \vec{F}\cdot\vec{n}\,dS = \iint_{\sigma_1} \vec{F}\cdot\vec{n}_1\,dS + \iint_{\sigma_2} \vec{F}\cdot\vec{n}\,dS + \iint_{\sigma_3} \vec{F}\cdot\vec{n}\,dS.$$

A normal a σ_1 é $\vec{n}_1 = \dfrac{u}{\sqrt{4-u^2-v^2}}\vec{i} + \dfrac{v}{\sqrt{4-u^2-v^2}}\vec{j} + \vec{k}$,

a normal a σ_2 é $\vec{n}_2 = \dfrac{u}{\sqrt{4-u^2-v^2}}\vec{i} + \dfrac{v}{\sqrt{4-u^2-v^2}}\vec{j} - \vec{k}$ e a

normal a σ_3 é $\vec{n}_3 = \cos u\vec{i} + \operatorname{sen} u\vec{j}$. Segue

$$\iint_{\sigma_1} \vec{F}\cdot\vec{n}_1\,dS = \iint_K (u\vec{i} + v\vec{j} + \sqrt{4-u^2-v^2}\,\vec{k})\cdot$$

$$\cdot\left(\frac{u}{\sqrt{4-u^2-v^2}}\vec{i} + \frac{v}{\sqrt{4-u^2-v^2}}\vec{j} + \vec{k}\right)du\,dv =$$

$$= \iint_K \frac{4}{\sqrt{4-u^2-v^2}}\,du\,dv, \text{ onde } K \text{ é o círculo } u^2+v^2 \leq 1.$$

Temos

$$\iint_{\sigma_1} \vec{F}\cdot\vec{n}_1\,dS = \int_0^{2\pi}\int_0^1 4(4-\rho^2)^{-\frac{1}{2}}\,\rho\,d\rho\,d\theta$$

$$= (-4)\int_0^{2\pi}\left[(4-\rho^2)^{1/2}\right]_0^1 d\theta = 8\pi(2-\sqrt{3}).$$

Analogamente, $\iint_{\sigma_2} \vec{F}\cdot\vec{n}_2\,dS = 8\pi(2-\sqrt{3})$

O fluxo de $\vec{F}$ através de σ_1 na direção $\vec{n}_1$ é $8\pi(2-\sqrt{3})$ e é igual ao fluxo de $\vec{F}$ na direção $\vec{n}_2$, através de σ_2.

O fluxo de $\vec{F}$, através de σ_3, na direção $\vec{n}_3$ é

$$\iint_{\sigma_3} \vec{F}\cdot\vec{n}_3\,dS = \iint_K (\cos u\vec{i} + \operatorname{sen} u\vec{j} + v\vec{k})\cdot(\cos u\vec{i} + \operatorname{sen} u\vec{j})\,du\,dv =$$

$$= \iint_K du\,dv = \int_0^{2\pi}\int_{-\sqrt{3}}^{\sqrt{3}} dv\,du = 4\pi\sqrt{3}.$$

Portanto,

$$\iint_\sigma \vec{F}\cdot\vec{n}\,dS = 8\pi(2-\sqrt{3}) + 8\pi(2-\sqrt{3}) + 4\pi\sqrt{3} = 32\pi - 12\pi\sqrt{3} \quad ①$$

Aplicando o teorema da divergência,

$$\iiint_B \underbrace{\operatorname{div}\vec{F}}_{3}\,dx\,dy\,dz = 3\iint_K \left[\int_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}} dz\right]dx\,dy =$$

$$= 6\iint_K \sqrt{4-x^2-y^2}\,dx\,dy = 6\int_0^{2\pi}\int_0^1 (4-\rho^2)^{1/2}\,\rho\,d\rho\,d\theta =$$

$$= \frac{6}{(-2)}\cdot\frac{2}{3}\int_0^{2\pi}\underbrace{\left[(4-\rho^2)\right]_0^1}_{(3\sqrt{3}-8)} d\theta = 4\pi(8-3\sqrt{3}) = 32\pi - 12\pi\sqrt{3} \quad ②$$

De ① e ② segue $\iint_\sigma \vec{F}\cdot\vec{n}\,dS = \iiint_B \operatorname{div}\vec{F}\,dx\,dy\,dz$.

11. Seja um escoamento com velocidade $\vec{v}(x, y, z)$ e densidade $\rho(x, y, z)$, tal que $\vec{w} = \rho\vec{v} = x\vec{i} + y\vec{j} - 2z\vec{k}$.

Consideremos $\sigma(u, v) = (u, v, \sqrt{4-u^2-v^2})$, $x^2+y^2 \leq 2$.

Utilizando o resultado do exercício anterior, o fluxo de $\vec{u}$ através de σ é

$$\iint_\sigma \vec{u}\cdot\vec{n}\,dS = \iint_K \left[P\frac{\partial f}{\partial x}(x,y) - Q\frac{\partial f}{\partial y}(x,y) + R\right] dx\,dy$$

onde $P = x$, $Q = y$, $R = -2\sqrt{4-x^2-y^2}$ e $z = f(x,y) = \sqrt{4-x^2-y^2}$.

Então,

$$\iint_\sigma \vec{u}\cdot\vec{n}\,dS = \iint_K \frac{-8+3(x^2+y^2)}{\sqrt{4-x^2-y^2}}\,dx\,dy, \text{ onde } K \text{ é o círculo } x^2+y^2 \leqslant 2.$$

Passando para coordenadas polares, tem-se $\iint_\sigma \vec{u}\cdot\vec{n}\,dS = -4\sqrt{2}\,\pi$.

Logo, o fluxo de $\vec{u}$ através de σ é $-4\sqrt{2}\,\pi$.

12. Sejam $\vec{u} = x\vec{i} + y\vec{j} - 2z\vec{k}$ e B o compacto $x^2+y^2+z^2 \leqslant 4$, $z \geqslant \sqrt{2}$.
Seja σ_1 a superfície $\sigma_1(u,v) = (u,v,\sqrt{2})$, $u^2+v^2 \leqslant 2$.

A fronteira de B coincide com a imagem da cadeia (σ_1, σ_2), onde
$\sigma_2(u,v) = (u,v,\sqrt{4-u^2-v^2})$, $u^2+v^2 \leqslant 2$.
Portanto, o fluxo de $\vec{u}$ através de σ é

$$\iint_\sigma \vec{u}\cdot\vec{n}\,dS = \iint_{\sigma_1} \vec{u}\cdot\vec{n}_1\,dS + \iint_{\sigma_2} \vec{u}\cdot\vec{n}_2\,dS.$$

A normal a σ_1 é $\vec{n}_1 = -\vec{k}$ e a normal a σ_2 é

$$\vec{n}_2 = \vec{k} + \frac{u}{\sqrt{4-u^2-v^2}}\vec{i} + \frac{v}{\sqrt{4-u^2-v^2}}\vec{j}$$

Temos

$$\iint_{\sigma_1} \vec{u}\cdot\vec{n}_1\,dS = \iint_K (u\vec{i}+v\vec{j}-2\sqrt{2}\,\vec{k})\cdot(-\vec{k})\,du\,dv = \iint_K 2\sqrt{2}\,du\,dv =$$

$$= \int_0^{2\pi}\int_0^{\sqrt{2}} 2\sqrt{2}\,\rho\,d\rho\,d\theta = 4\sqrt{2}\,\pi.$$

$$\iint_{\sigma_2} \vec{u}\cdot\vec{n}_2\,dS = \iint_K (u\vec{i}+v\vec{j}-2\sqrt{4-u^2-v^2}\,\vec{k})\cdot$$

$$\cdot\left(+\frac{u\vec{i}}{\sqrt{4-u^2-v^2}} + \frac{v\vec{j}}{\sqrt{4-u^2-v^2}} + \vec{k}\right) du\,dv =$$

$$= \iint_k \frac{3u^2+3v^2-8}{\sqrt{4-u^2-v^2}}\,du\,dv = -4\sqrt{2}\,\pi. \text{ (Veja o exercício anterior.)}$$

Portanto, $\iint_\sigma \vec{u}\cdot\vec{n}\,dS = 4\sqrt{2}\pi - 4\sqrt{2}\pi = 0$.

Por outro lado, temos $\operatorname{div}\vec{u} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = \frac{\partial(-2z)}{\partial z} = -2$ e $\iiint_B \underbrace{\operatorname{div}\vec{u}}_{0}\,dx\,dy\,dz = 0$.

Logo, $\iint_\sigma \vec{u}\cdot\vec{n}\,dS = \iiint_B \operatorname{div}\vec{u}\,dx\,dy\,dz$.

13. Pelo teorema da divergência

$$\iint_\sigma \vec{u}\cdot\vec{n}\,dS = \iiint_B \operatorname{div}\vec{u}\,dx\,dy\,dz = 0.$$

Exercícios 10.2

1. Pelo teorema da divergência, temos

$$\iint_\sigma \operatorname{rot}\vec{u}\cdot\vec{n}\,dS = \iiint_B (\operatorname{div}\operatorname{rot}\vec{u})\,dx\,dy\,dz.$$

Mas div rot $\vec{u} = 0$. (Veja Seção 2.4, Exercício 9(e), deste livro.)

Logo, $\iint_\sigma \operatorname{rot}\vec{u}\cdot\vec{n}\,dS = 0$.

3. Seja $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$.

Pelo teorema da divergência,

$$\frac{1}{3}\iint_\sigma \vec{r}\cdot\vec{n}\,dS = \frac{1}{3}\iiint_B \underbrace{(\operatorname{div}\vec{r})}_{3}\,dx\,dy\,dz =$$

$$= \iiint_B dx\,dy\,dz = \text{volume } B.$$

4. Sejam B o compacto $x^2+y^2+z^2 \leqslant 1$, $z \geqslant 0$ (semi-esfera), σ a superfície $x^2+y^2+z^2 = 1$, $z > 0$ e σ, dada por $\sigma_1(u,v) = (u,v,0)$, $u^2+v^2 \leqslant 1$.

A fronteira de B coincide com a imagem da cadeia (σ, σ_1). Pelo teorema da divergência, temos

$$\iint_\sigma \vec{F}\cdot\vec{n}\,dS + \iint_{\sigma_1} \vec{F}\cdot(-\vec{k})\,dS = \iiint_B \operatorname{div}\vec{F}\,dx\,dy\,dz \quad ①$$

De div $\vec{F} = y^2 + xz + 1 - y^2 - xz = 1$, segue que

$$\iiint_B \operatorname{div} \vec{F}\, dx\, dy\, dz = \iiint_B dx\, dy\, dz = \frac{2}{3}\pi. \quad ②$$

Vamos calcular o fluxo de $\vec{F}$, através de σ_1, na direção $(-\vec{k})$.

$\iint_{\sigma_1} \vec{F}\cdot(-\vec{k})\, dS = \iint_{\sigma_1} (y^2 z + \frac{1}{2} xz^2 - z)\, dS = 0$, pois

$(x, y, z) \in \operatorname{Im} \sigma_1$ e, portanto, $z = 0$.

Então, $\iint_{\sigma_1} \vec{F}\cdot(-\vec{k})\, dS = 0$ ③

Substituindo ② e ③ em ①, segue:

$$\iint_\sigma \vec{F}\cdot\vec{n}\, dS = \frac{2}{3}\pi.$$

7. *a*) Seja $B = \{(x, y, z) \in \mathbb{R}^3 \mid 0 \leq x \leq 1, 0 \leq y \leq x \text{ e } 0 \leq z \leq 4\}$ e
$\vec{u} = xy\vec{i} + yz\vec{j} + z^2\vec{k}$

Aplicando o teorema da divergência,

$$\iint_\sigma \vec{u}\cdot\vec{n}\, dS = \iiint_B \operatorname{div} \vec{u}\, dx\, dy\, dz = \iiint_B (y + 3z)\, dz\, dx\, dy =$$
$$= \int_0^1 \int_0^x \left[\int_0^4 (y + 3z)\, dz\right] dy\, dx = \frac{38}{3}.$$

b) Seja $B = \{(x, y, z) \in \mathbb{R}^3 \;\; x^2 + y^2 + z^2 \leq 1 \text{ e } z \geq x + y\}$ e
$\vec{u} = -2xy\vec{i} + y^2\vec{j} + 3z\vec{k}$. Observe que B é uma semi-esfera de raio 1, logo, o volume de B é $\frac{2\pi}{3}$.

$$\iint_\sigma \vec{u}\cdot\vec{n}\, dS = \iiint_B (\operatorname{div} \vec{u})\, dx\, dy\, dz = 3\iiint_B dx\, dy\, dz = 2\pi.$$

d) Seja $B = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 \leq 1, x^2 + y^2 \leq z \leq 5 - x^2 - y^2\}$ e
$\vec{u} = 3xy\vec{i} - \frac{3}{2} y^2\vec{j} + z\vec{k}$.

Temos

$$\iint_\sigma \vec{u}\cdot\vec{n}\, dS = \iiint_B (\operatorname{div} \vec{u})\, dx\, dy\, dz = \iiint_B dz\, dx\, dy = 4\pi.$$

e) Seja B o paralelepípedo $0 \leq x \leq 1, 0 \leq y \leq 1$ e $0 \leq z \leq 1$ e
$\vec{u} = x^3\vec{i} + y^3\vec{j} + z^3\vec{k}$

Temos

$$\iint_\sigma \vec{u}\cdot\vec{n}\, dS = \iiint_B \underbrace{(3x^2 + 3y^2 + 3z^2)}_{\operatorname{div} \vec{u}}\, dx\, dy\, dz$$
$$= \iint_K \left[\int_0^1 (3x^2 + 3y^2 + 3z^2)\, dz\right] dx\, dy, \text{ onde } K \text{ é o}$$

retângulo $0 \leq x \leq 1, 0 \leq y \leq 1$.

Segue que

$$\iint_K \left[3x^2 z + 3y^2 z + z^3\right]_0^1 dx\, dy = \int_0^1 \int_0^1 (3x^2 + 3y^2 + 1)\, dx\, dy =$$
$$= \int_0^1 (2 + 3y^2)\, dy = 3.$$

8. Sejam $\sigma(u, v) = (u, v, u^2 + v^2)$, $u^2 + v^2 \leq 1$, $\vec{n}$ a normal a σ com componente $z \leq 0$ e $\vec{F}(x, y, z) = x^2 y\,\vec{i} - xy^2\,\vec{j} + \vec{k}$.

Temos

$\frac{\partial \sigma}{\partial u} \wedge \frac{\partial \sigma}{\partial v} = -2u\vec{i} - 2v\vec{j} + \vec{k}$. Consideremos $\vec{n} = \frac{2u\vec{i} + 2v\vec{j} - \vec{k}}{\sqrt{1 + 4u^2 + 4v^2}}$.

Temos

$$\iint_\sigma \vec{F}\cdot\vec{n}\, dS = \iint_k (u^2 v\vec{i} - uv^2\vec{j} + \vec{k})\cdot(2u\vec{i} + 2v\vec{j} - \vec{k})\, du\, dv =$$
$$= \iint_K \left[2uv(u^2 - v^2) - 1\right] du\, dv.$$

Passando para coordenadas polares

$$\iint_\sigma \vec{F}\cdot\vec{n}\, dS = -\int_0^{2\pi} \left[\int_0^1 (1 - 2\rho^4 \operatorname{sen}\theta \cos\theta (\cos^2\theta - \operatorname{sen}^2\theta))\, \rho\, d\rho\right] d\theta = -\pi.$$

9. Como $\vec{u}$ é de classe C^1 em Ω, div $\vec{u}$ é contínua em Ω. Suponhamos que exista $X_0 \in \Omega$ tal que div $\vec{u}$ $(X_0) \neq 0$. Para fixar o raciocínio, suporemos div $\vec{u}$ $(X_0) > 0$. Pela conservação do sinal existirá uma bola B fechada, de centro X_0, que podemos supor contida em Ω, pois Ω é aberto. Sendo σ a fronteira de B, com normal exterior $\vec{n}$, teremos

$$\iint_\sigma \vec{u}\cdot\vec{n}\, dS = \iiint_B \operatorname{div} \vec{u} > 0$$

contra a hipótese. Logo, div $\vec{u} = 0$ em Ω.

10. Sejam $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ e $r = \|\vec{r}\|$. Então, $r = \sqrt{x^2 + y^2 + z^2}$.

Façamos $\vec{F} = \frac{\vec{r}}{r^3} = \frac{x}{r^3}\vec{i} + \frac{y}{r^3}\vec{j} + \frac{z}{r^3}\vec{k}$.

$\vec{F}$ é de classe C^1 em $\mathbb{R}^3 - \{(0, 0, 0)\}$.

Seja σ a fronteira da esfera K, com normal exterior $\vec{n}$. Suponhamos que a origem não pertença a K. Pelo teorema da divergência,

$$\iint_\sigma \frac{\vec{r}}{r^3} \cdot \vec{n}\, dS = \iiint_K \operatorname{div}\left(\frac{\vec{r}}{r^3}\right) dx\, dy\, dz =$$

$$= \iiint_K \left(\frac{\partial}{\partial x}(x \cdot r^{-3}) + \frac{\partial}{\partial y}(y \cdot r^{-3}) + \frac{\partial}{\partial z}(z \cdot r^{-3})\right) dx\, dy\, dz =$$

$$= \iiint_K (r^{-3} - 3x^2 r^{-5} + r^{-3} - 3y^2 r^{-5} + r^{-3} - 3z^2\, r^{-5})\, dx\, dy\, dz =$$

$$= \iiint_K \left(3r^{-3} - 3\underbrace{(x^2 + y^2 + z^2)}_{r^2} r^{-5}\right) dx\, dy\, dz = 0.$$

Seja σ_1 a fronteira, com normal exterior $\vec{n}_1$, de uma esfera B, com centro na origem e raio R, contida no interior de K. Seja K_1 a região limitada por σ e σ_1. Em K_1, o teorema da divergência se aplica e temos

$$\iint_\sigma \vec{F} \cdot \vec{n}\, dS - \iint_{\sigma_1} \vec{F} \cdot \vec{n}_1\, dS = \iiint_{K_1} \operatorname{div} F\, dx\, dy\, dz = 0, \text{ daí}$$

$$\iint_\sigma \vec{F} \cdot \vec{n}\, dS = \iint_{\sigma_1} \vec{F} \cdot \vec{n}_1\, dS.$$

De $\vec{F} \cdot \vec{n} = \frac{\vec{r}}{r^3} \cdot \frac{\vec{r}}{r} = \frac{(x\vec{i} + y\vec{j} + z\vec{k})}{r^3} \frac{(x\vec{i} + y\vec{j} + z\vec{k})}{r} = \frac{1}{r^2}$

segue $\iint_{\sigma_1} \vec{F} \cdot \vec{n}\, dS = \frac{1}{R^2} \iint_{\sigma_1} dS = \frac{4\pi R^2}{R^2} = 4\pi$.

Logo, $\iint_\sigma \vec{F} \cdot \vec{n}\, dS = 4\pi$.

CAPÍTULO 11

Exercícios 11.1

1. *a*) Consideremos $F(x, y, z) = y\vec{k}$. Sejam $\sigma(u, v) = (u, v, u^2 + v^2)$, $u^2 + v^2 \leq 1$, e $\vec{n}$ a normal apontando para cima.

Pelo teorema de Stokes: $\iint_\sigma \operatorname{rot} \vec{F} \cdot \vec{n}\, ds = \int_\Gamma \vec{F} \cdot d\vec{r}$.

Seja $\Gamma(t) = \sigma(\cos t, \operatorname{sen} t) = (\cos t, \operatorname{sen} t, 1)$, $t \in [0, 2\pi]$.

Temos $\int_\Gamma \vec{F} \cdot d\vec{r} = \int_0^{2\pi} (\operatorname{sen} t\, \vec{k}) \cdot (-\operatorname{sen} t\, \vec{i} + \cos t\, \vec{j})\, dt = 0$.

Portanto, $\iint_\sigma \operatorname{rot} \vec{F} \cdot \vec{n}\, ds = 0$.

c) Consideremos $\vec{F}(x, y, z) = y\vec{i} + x^2\vec{j} + z\vec{k}$ e
$\sigma(u, v) = (u, v, 2u + v = 1)$, $u \geq 0$, $v \geq 0$ e $u + v \leq 2$.
$\vec{n}$ é a normal apontando para baixo.

Temos

$\frac{\partial \sigma}{\partial u} \wedge \frac{\partial \sigma}{\partial v} = -2\vec{i} - \vec{j} + \vec{k}$. Então, $\vec{n}_1 = -\frac{2\vec{i}}{\sqrt{6}} - \frac{\vec{j}}{\sqrt{6}} + \frac{\vec{k}}{\sqrt{6}}$ é a normal apontando para cima, pois a componente de $\vec{k}$ é positiva.

Segue que $\vec{n} = -\vec{n}_1$

Logo, $\iint_\sigma \operatorname{rot} \vec{F} \cdot \vec{n}\, ds = -\iint_\sigma \operatorname{rot} \vec{F} \cdot \vec{n}_1\, ds$.

Vamos calcular $\iint_\sigma \operatorname{rot} \vec{F} \cdot \vec{n}_1\, ds$ aplicando Stokes:

Seja

$\gamma_1(t) = (t, 0)$, $0 \leq t \leq 2$; $\gamma_2(t) = (2 - t, t)$, $0 \leq t \leq 2$; $\gamma_3(t) = (0, 2 - t)$, $0 \leq t \leq 2$

Segue que

$\Gamma_1(t) = \sigma(\gamma_1(t)) = (t, 0, 2t + 1), 0 \leq t \leq 2$

$\Gamma_2(t) = \sigma(\gamma_2(t)) = (2 - t, t, 5 - t), 0 \leq t \leq 2$

$\Gamma_3(t) = \sigma(\gamma_3(t)) = (0, 2 - t, 3 - t), 0 \leq t \leq 2.$

Então,

$$\int_{\Gamma_1} \vec{F} \cdot d\vec{r} = \int_0^2 (t^2\vec{j} + (2t+1)\vec{k}) \cdot (\vec{i} + 2\vec{k})\, dt = \int_0^2 (4t+2)\, dt = 12$$

$$\int_{\Gamma_2} \vec{F} \cdot d\vec{r} = \int_0^2 (t\vec{i} + (2-t)^2\vec{j} + (5-t)\vec{k}) \cdot (-\vec{i} + \vec{j} - \vec{k})\, dt = -\frac{22}{3}$$

$$\int_{\Gamma_3} \vec{F} \cdot d\vec{r} = \int_0^2 ((2-t)\vec{i} + (3-t)\vec{k}) \cdot (-\vec{j} - \vec{k})\, dt = -4$$

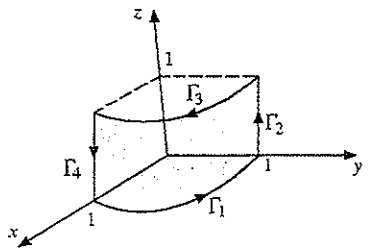
Logo,

$$\int_\Gamma \vec{F} \cdot d\vec{r} = 12 - \frac{22}{3} - 4 = \frac{2}{3} \text{ e } \iint_\sigma \text{rot } \vec{F} \cdot \vec{n}_1\, ds = \frac{2}{3}.$$

Portanto, $\iint_\sigma \text{rot } \vec{F} \cdot \vec{n}\, ds = -\frac{2}{3}$.

e) Consideremos $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$, onde $P = 0$, $Q = x$ e $R = 0$. Temos

$$\iint_\sigma \text{rot } \vec{F} \cdot \vec{n}\, dS = \int_\Gamma x\, dy.$$



$$\Gamma_1: \begin{cases} x = \cos t \\ y = \text{sen } t \\ z = 0 \end{cases} \quad 0 \leq t \leq \frac{\pi}{2}, \qquad \Gamma_2: \begin{cases} x = 0 \\ y = 1 \\ z = t \end{cases} \quad 0 \leq t \leq 1$$

$$\Gamma_3: \begin{cases} x = \cos t \\ y = \text{sen } t \\ z = 1 \end{cases} \quad 0 \leq t \leq \frac{\pi}{2} \text{ e } \quad \Gamma_4: \begin{cases} x = 1 \\ y = 0 \\ z = 1 - t \end{cases} \quad 0 \leq t \leq 1.$$

De $dy = \cos t\, dt$ (Γ_1), $dy = 0$ (Γ_2), $dy = \cos t\, dt$ (Γ_3) e $dy = 0$ (Γ_4) resulta

$$\int_\Gamma x\, dy = \int_0^{\frac{\pi}{2}} \cos^2 t\, dt - \int_0^{\frac{\pi}{2}} \cos^2 t\, dt = 0.$$

Portanto, $\iint_\sigma \text{rot } \vec{F} \cdot \vec{n}\, dS = 0$.

Por outro lado, podemos calcular $\iint_\sigma \text{rot } \vec{F} \cdot \vec{n}\, dS$ diretamente. Temos

$$\text{rot } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & x & 0 \end{vmatrix} = \vec{k}.$$

Sendo $\sigma(u, v) = (\cos u, \text{sen } u, v)$, $0 \leq v \leq 1$ temos

$$\frac{\partial \sigma}{\partial u} \wedge \frac{\partial \sigma}{\partial v} = \cos u\, \vec{i} + \text{sen } u\, \vec{j} = \vec{n}.$$

Então, rot $\vec{F} \cdot \vec{n} = 0$ e daí

$$\iint_\sigma \text{rot } \vec{F} \cdot \vec{n}\, dS = 0.$$

g) Consideremos $\vec{F}(x, y, z) = y\vec{i}$ e $\sigma(u, v) = (u, v, \sqrt{2 - u^2 - v^2})$, $u^2 + v^2 \leq 1$.
Seja $\Gamma(t) = (\cos t, \text{sen } t, 1)$ a curva fronteira de σ orientada positivamente em relação à normal $\vec{n}$.

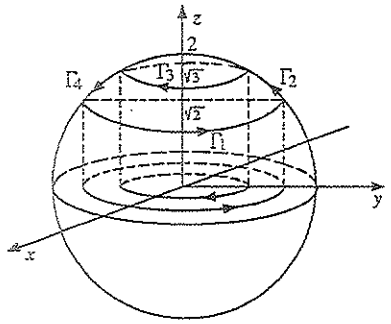
Temos

$$\iint_\sigma \text{rot } \vec{F} \cdot \vec{n}\, dS = \int_\Gamma \vec{F}\, d\vec{r} = \int_0^{2\pi} (\text{sen } t\, \vec{i})(-\text{sen } t\, \vec{i} + \cos t\, \vec{j})\, dt =$$

$$= \int_0^{2\pi} (-\text{sen}^2 t)\, dt = -\pi.$$

h) Consideremos $\vec{F}(x, y, z) = -y\vec{i} + x\vec{j} + x^2\vec{k}$ e σ a superfície $x^2 + y^2 + z^2 = 4$, $\sqrt{2} \leq z \leq \sqrt{3}$ e $y \geq 0$.

Temos $\sigma(u, v) = (u, v, \sqrt{4 - u^2 - v^2}), \sqrt{2} \leq \sqrt{4 - u^2 - v^2} \leq \sqrt{3}, v \geq 0$
ou seja, $1 \leq u^2 + v^2 \leq 2, v \geq 0$
Vamos considerar as integrais sobre as curvas Γ_1, $\bar{\Gamma}_2$, $\bar{\Gamma}_3$ e Γ_4, assim definidas



$\Gamma_1(t) = (\sqrt{2}\cos t, \sqrt{2}\,\text{sen}\, t, \sqrt{2}), t \in [0, \pi]$,
$\bar{\Gamma}_3(t) = (\cos t, \text{sen}\, t, \sqrt{3}), t \in [0, \pi]$,
$\Gamma_4(t) = (t, 0, \sqrt{4 - t^2}, 1 \leq t \leq 2$, e
$\Gamma_2(t) = (t, 0, \sqrt{4 - t^2},), -2 \leq t \leq -1$.

Segue que

$$\int_{\Gamma_1} \bar{F}\, d\bar{r} = \int_0^{\pi} (-\sqrt{2}\,\text{sen}\, t\, \bar{i} + \sqrt{2}\cos t\, \bar{j} + 2\cos^2 t\, \bar{k}) \cdot$$
$$\cdot (-\sqrt{2}\,\text{sen}\, t\, \bar{i} + \sqrt{2}\cos t\, \bar{j})\, dt = \int_0^{\pi} 2\, dt = 2\pi,$$

$$\int_{\bar{\Gamma}_3} \bar{F}\, d\bar{r} = \int_0^{\pi} (-\text{sen}\, t\, \bar{i} + \cos t\, \bar{j} + \cos^2 t\, \bar{k})(-\text{sen}\, t\bar{i} + \cos t\bar{j})\, dt =$$
$$= \int_0^{\pi} dt = \pi \text{ e}$$

$\int_{\Gamma_2} \bar{F} \cdot d\bar{r} = -\int_{\Gamma_4} \bar{F} \cdot d\bar{r}$ (verifique).

Como $\int_{\Gamma_3} \bar{F} \cdot d\bar{r} = -\int_{\bar{\Gamma}_3} \bar{F} \cdot d\bar{r}$, resulta

$$\iint_{\sigma} \text{rot}\, \bar{F} \cdot \bar{n} ds = \int_{\Gamma_1} \bar{F} \cdot d\bar{r} + \int_{\Gamma_2} \bar{F} \cdot d\bar{r} + \int_{\Gamma_3} \bar{F} \cdot d\bar{r} + \int_{\Gamma_4} \bar{F} \cdot d\bar{r} = \pi.$$

j) Sejam $\bar{F}(x, y, z) = y\bar{i} + x\bar{j} + xz\bar{k}$ e σ a superfície $z = x + y + 2$ e $x^2 + \frac{y^2}{4} \leq 1$.

Façamos $\sigma(u, v) = (u, v, u + v + 2)$, $u^2 + \frac{v^2}{4} \leq 1$.

$\frac{\partial \sigma}{\partial u} \wedge \frac{\partial \sigma}{\partial v} = -\bar{i} - \bar{j} + \bar{k}$. Como a componente de $\bar{k}$ é positiva temos que a normal

$\bar{n}_1 = \frac{\frac{\partial \sigma}{\partial u} \wedge \frac{\partial \sigma}{\partial v}}{\left\| \frac{\partial \sigma}{\partial u} \wedge \frac{\partial \sigma}{\partial v} \right\|}$ aponta para cima. Seja $\bar{n}$ a normal que aponta para baixo. Segue que

$$\iint_{\sigma} \text{rot}\, \bar{F} \cdot \bar{n}\, dS = -\iint_{\sigma} \text{rot}\, \bar{F} \cdot \bar{n}_1\, dS.$$

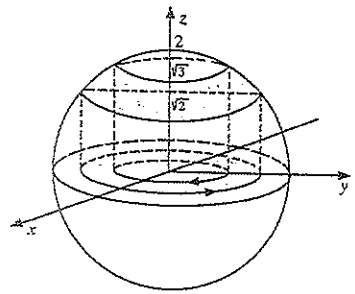
Seja $\Gamma(t) = (\cos t, 2\,\text{sen}\, t, \cos t + 2\,\text{sen}\, t + 2)$, $t \in [0, 2\pi]$, a curva fronteira de σ. Pelo Teorema de Stokes,

$$\iint_{\sigma} \text{rot}\, \bar{F} \cdot \bar{n}_1\, dS = \int_{\Gamma} \bar{F}\, d\bar{r} =$$
$$= \int_0^{2\pi} (2\,\text{sen}\, t\, \bar{i} + \cos t\, \bar{j} + (\cos^2 t + 2\,\text{sen}\, t\cos t + 2\cos t)\, \bar{k}) \cdot$$
$$\cdot (-\text{sen}\, t\, \bar{i} + 2\cos t\, \bar{j} + (2\cos t - \text{sen}\, t)\, \bar{k})\, dt =$$
$$= \int_0^{2\pi} (-2\,\text{sen}^2 t + 2\cos^2 t + (2\cos t - \text{sen}\, t)(\cos^2 t + 2\,\text{sen}\, t\cos t +$$
$$+ 2\cos t)]\, dt = 4\pi.$$

Assim, $\iint_{\sigma} \text{rot}\, \bar{F} \cdot \bar{n} dS = -4\pi$.

4. Consideremos $\bar{F}(x, y, z) = xz^2\bar{i} + z^4\bar{j} + yz\bar{k}$ e
$\sigma(u, v) = (u, v, \sqrt{4 - u^2 - v^2}), 1 \leq u^2 + v^2 \leq 2$

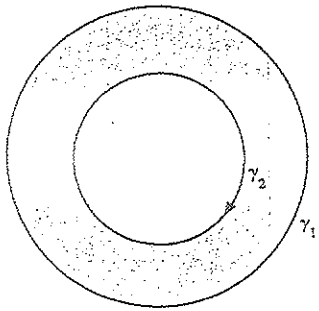
Sejam $\Gamma_1(t) = \sigma(\gamma_1(t)) = (\sqrt{2}\cos t, \sqrt{2}\,\text{sen}\, t, \sqrt{2})$, $t \in [0, 2\pi]$ e
$\Gamma_2(t) = \text{o}\,(\gamma_2(t)) = (\cos t, \text{sen}\, t, \sqrt{3}), t \in [0, 2\pi]$ sendo γ_1, γ_2 curvas simples fechadas (γ_1 orientada no sentido anti-horário e γ_2 no sentido horário).



Temos

$$\int_{\Gamma_1} \bar{F}\, d\bar{r} = \int_0^{2\pi} (2\sqrt{2}\cos t\,\bar{i} + 4\bar{j} + 2\,\text{sen}\, t\,\bar{k})(-\sqrt{2}\,\text{sen}\, t\,\bar{i} + \sqrt{2}\cos t\,\bar{j})\, dt$$

$$= \int_0^{2\pi} (-4\,\text{sen}\, t\cos t + 4\sqrt{2}\cos t)\, dt = 0;$$



$$\int_{\Gamma_2} \bar{F}\, d\bar{r} = \int_0^{2\pi} (3\cos t\,\bar{i} + 9\bar{j} + \sqrt{3}\,\text{sen}\, t\,\bar{k})(-\text{sen}\, t\,\bar{i} + \cos t\,\bar{j})\, dt =$$

$$= \int_0^{2\pi} (-3\,\text{sen}\, t\cos t + 9\cos t)\, dt = 0.$$

Pelo exercício anterior, $\iint_\sigma \text{rot}\,\bar{F}\cdot\bar{n}\, dS = \int_{\Gamma_1} \bar{F}\, d\bar{r} + \int_{\Gamma_2} \bar{F}\, d\bar{r} = 0.$

5. Consideremos $\bar{F}(x, y, z) = x^3\bar{k}$ e

$$\sigma(u, v) = (u, v, v + 4),\ 1 \leq u^2 + v^2 \leq 4.$$

Temos

$$\frac{\partial\sigma}{\partial u} \wedge \frac{\partial\sigma}{\partial v} = -\bar{j} + \bar{k} \text{ e } \text{rot}\,\bar{F} = 3x^2\bar{j}.$$

Segue que

$$\iint_\sigma \text{rot}\,\bar{F}\cdot\bar{n}\, dS = \iint_\sigma (3u^2\bar{j})\cdot(-\bar{j} + \bar{k})\, du\, dv =$$

$$= \iint_\sigma 3u^2\, du\, dv = 3\int_0^{2\pi}\int_1^2 \rho^2\cos^2\theta\,\rho\, d\rho\, d\theta =$$

$$= 3\int_0^{2\pi}\left[\frac{\rho^4}{4}\right]_1^2 \cos^2\theta\, d\theta = 3\cdot\frac{15}{4}\cdot\frac{1}{2}\int_0^{2\pi}(1 + \cos 2\theta)\, d\theta =$$

$$= \frac{45}{4}\cdot\frac{1}{2}\left[\theta + \frac{\text{sen}\, 2\theta}{2}\right]_0^{2\pi} = \frac{45\pi}{4}.$$